Technische Universität Kaiserslautern

Crash Hedging Strategies and Optimal Portfolios

© Olaf Arnd Menkens

Vom Fachbereich Mathematik
der Technischen Universität Kaiserslautern
zur Verleihung des akademischen Grades
Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.)
genehmigte Dissertation

1. Gutachter: Prof. Dr. Ralf Korn

2. Gutachter: Prof. Dr. Mogens Steffensen

Datum der Disputation: 30.11.2004

D 386

Abstract

In traditional portfolio optimization under the threat of a crash the *investment horizon* or the *time to maturity* is neglected.

Developing the so–called *crash hedging strategies* (which are portfolio strategies which make an investor indifferent to the occurrence of an uncertain (down) jumps of the price of the risky asset) the time to maturity turns out to be essential. The crash hedging strategies are derived as solutions of non–linear differential equations which itself are consequences of a equilibrium strategy. Hereby the situation of changing market coefficients after a possible crash is considered for the case of logarithmic utility as well as for the case of general utility functions. A benefit–cost analysis of the crash hedging strategy is done as well as a comparison of the crash hedging strategy with the optimal portfolio strategies given in traditional crash models.

Moreover, it will be shown that the crash hedging strategies optimize the worst–case bound for the expected utility from final wealth subject to some restrictions. Another application is to model crash hedging strategies in situations where both the number and the height of the crash are uncertain but bounded. Taking the additional information of the probability of a possible crash happening into account leads to the development of the *q–quantile crash hedging strategy.*

Keywords: Portfolio optimization, crash modelling, crash hedging, Hamilton–Jacobi–Bellman equation, equilibrium strategies, worst-case scenario, changing market coefficients, q–quantile crash hedging strategy.

Contents

1 Introduction

A market crash is a synonym of a worst–case scenario for an investor trading in a security market. Therefore, to be prepared for such a situation is a desirable goal. One can of course do this by buying suitable put options. However, being in such a well–insured situation is quite expensive. In fact, one then really needs the crash such that the purchase of these options will not be in vain. In contrast to this, the approach presented in this thesis shows that it is possible to be indifferent on the occurrence or non–occurrence of a crash *without* the help of additional derivatives, just by following a suitable investment strategy in bond and stock.

Modelling of a crash or – more general – of large stock price movements is an actively researched field in financial mathematics (see e.g. Aase [1], Merton [12], Eberlein and Keller [4], or Embrechts, Klüppelberg and Mikosch [5] just as representatives for various sources). Mostly those approaches rely on modelling stock prices as Lévy processes or other types of processes with heavy–tailed distributions. Using a different approach to that, in this thesis the view will be taken of a semi–specialized stock price process. More precisely, the distinction will be made between so–called "normal times" where the stock prices are assumed to follow a geometric Brownian Motion and "crash times" where the stock price falls suddenly.

This approach puts the emphasis on

- avoiding large losses in *any* possible situation by maximizing the worst–case bound for the utility of terminal wealth.
- the *investment horizon* or the time to maturity, which is very important in crash modelling. However, this variable is neglected in traditional portfolio optimization under the threat of a crash.
- the possible number of crashes within the investment horizon instead of the crash intensity in the traditional crash modelling. Moreover, only a range for the possible crash size is needed instead of a specific crash size.

This approach is already looked at in a paper by Korn and Wilmott [11] where the authors determined optimal portfolios under the threat of a crash in the case of logarithmic utility for final wealth. There, the main aim is to show that still suitable investment in stocks can be more profitable than playing safe and investing everything in the riskless bond if a crash of the stock price can occur. The corresponding optimal strategy is found via the solution of a balance problem between obtaining good worst–case bounds in case of a crash on one hand and also a reasonable performance on the other hand, if no crash occurs at all. The model has been extended to general utility functions in a recent paper by Korn and Menkens [10].

Using the approach of Korn and Wilmott [11] the aim of this thesis is to generalize the model in various direction and to scrutinize some of its properties. The most important aims are

- introducing the crash hedging strategy (see Definition 2.4).
- considering changing market conditions after a possible crash for logarithmic utility (see Chapter 2) as well as for general utility functions and where the worst case crash size may depend on both, time and wealth (see Chapter 3).
- proving the existence and uniqueness of a solution for the differential equations (13) and (77).
- calculating the cost and the potential benefit of the crash hedging strategy (see Section 2.6).
- taking the additional information of the probability of a possible crash happening into account by developing the q–quantile crash hedging strategy (see Section 2.13).
- giving a geometric interpretation of the crash hedging strategy (see Section 2.9).
- comparing this model with the traditional crash model (see Section 2.14).

Let us finally remark that the above approach is not limited to problems in financial mathematics but can also be applied to other areas of stochastic control applications where from time to time catastrophes may take place. This could open up a whole new field of stochastic control theory.

The thesis is organized as follows: Chapter 2 analyses the model in various directions for the logarithmic utility case. The main results of Chapter 2 are extended in Chapter 3 to general utility functions and to time– and wealth–depending worst case crash sizes. The last chapter gives a conclusion and an outlook.

Acknowledgement

At this point, I like to thank my wife Sabine and my daughter Anna Tabea for being so patient with me while I wrote my thesis.

Moreover, I thank Professor Ralf Korn for introducing this interesting subject to me and for supervising my thesis. I appreciated his comments and advices which improved my thesis considerably.

I have also benefited from discussions with Christian Ewald, Professor Mogens Steffensen, Jean–Pierre Stockis, and Professor Peter Imkeller.

2 The Logarithmic Utility Case

2.1 The Set Up

As in Korn and Wilmott [11], let us start with the most basic setting and consider a security market consisting of a riskless bond and a single risky security with prices given by

$$dP_{0,0}(t) = P_{0,0}(t)\, r_0\, dt\,, \qquad P_{0,0}(0) = 1\,, \tag{1}$$
$$dP_{0,1}(t) = P_{0,1}(t)\, [\mu_0\, dt + \sigma_0\, dW(t)]\,, \qquad P_{0,1}(0) = p_1\,, \tag{2}$$

with constant market coefficients μ_0, r_0 and $\sigma_0 \neq 0$ in "normal times" and where W is a Brownian Motion on a complete probability space $(\Omega, \mathcal{F}, P)$. Assume further that at most one crash can happen within the time horizon T. At the "crash time" the stock price suddenly falls. More specific, suppose that the sudden relative fall of the stock price lies in the interval $[k_*, k^*]$, where the constants $0 < k_* < k^* < 1$ ("the lowest and the highest possible crash size, respectively") are given. No probabilistic assumptions are made about the distribution of either the crash time or the crash height. The idea behind this approach is that practitioners often have difficulties in predicting jump intensities or other rates corresponding to large price movements. However, just specifying an upper bound for both the number of crashes and their heights seems to be an easier task.

Since it has been assumed that the investor is able to realize that the crash has happened let us model its occurrence via a jump process $N(t)$ which is zero before the jump time and equals one from the jump time onwards. Let us require that N lives also on $(\Omega, \mathcal{F}, P)$. To model the fact that the investor is able to realize that a jump of the stock price has happened it is supposed that the investor's decisions are adapted to the P-augmentation $\{\mathcal{F}_t\}$ of the filtration generated by both the Brownian motion $W(t)$ and the jump process $N(t)$.

Let us further suppose that the market conditions change after a possible crash. Let therefore k (with $k \in [k_*, k^*]$) be the arbitrary size of a crash at time τ. The price of the bond and the risky asset is assumed to be

$$dP_{1,0}(t) = P_{1,0}(t)\, r_1\, dt\,, \qquad P_{1,0}(\tau) = P_{0,0}(\tau)\,, \tag{3}$$
$$dP_{1,1}(t) = P_{1,1}(t)\, [\mu_1\, dt + \sigma_1\, dW(t)]\,, \qquad P_{1,1}(\tau) = (1-k)\, P_{0,1}(\tau)\,, \tag{4}$$

with constant market coefficients r_1, μ_1 and $\sigma_1 \neq 0$ after a possible crash of size k at time τ.

For simplicity, the initial market will also be called market 0, while the market after a crash will be called market 1.

It is important to keep in mind that the investor does *not* know that a crash will occur, the investor thinks only that it is possible. An investor who knows that a crash will happen within the time horizon $[0, T]$ has additional information and is therefore an insider. The set of possible crash heights of the insider is indeed

$K_I := [k_*, k^*]$, while the set of possible crash heights of the investor who thinks that a crash is possible is $K := \{0\} \cup [k_*, k^*]$. In this paper only the portfolio problem of the investor, who thinks a crash is possible, is considered.

Definition 2.1

1. *For $i = 0, 1$, let* $\boldsymbol{A_i(s,x)}$ *be the* **set of admissible portfolio processes** *$\pi(t)$ corresponding to an initial capital of $x > 0$ at time s, i.e. $\{\mathcal{F}_t, s \le t \le T\}$–progressively measurable processes such that*

   (i) the **wealth equation** *in market i in the usual crash–free setting*

$$dX_i^{\pi,s,x}(t) = X_i^{\pi,s,x}(t)\left[(r_i + \pi(t)[\mu_i - r_i])\,dt + \pi(t)\sigma_i\,dW_i(t)\right], \quad (5)$$

$$X_i^{\pi,s,x}(s) = x \quad (6)$$

   has a unique non–negative solution $X_i^{\pi,s,x}(t)$ and satisfies

$$\int_s^T \left[\pi(t)X_i^{\pi,s,x}(t)\right]^2\,dt < \infty \quad P\text{–a.s.}, \quad (7)$$

   i.e. $\boldsymbol{X_i^{\pi,s,x}(t)}$ *is the* **wealth process in market $\boldsymbol{i}$** *in the crash–free world, which uses the portfolio strategy π and starts at time s with initial wealth x.*
   Furthermore, $X_i^{\pi}(t) := X_i^{\pi,0,x}(t)$ will be used as an abbreviation.

   (ii) $\pi(t)$ has left–continuous paths with right limits.

2. *the corresponding* **wealth process** $\boldsymbol{X^\pi(t)}$ **in the crash model**, *defined as*

$$X^\pi(t) = \begin{cases} X_0^\pi(t) & \text{for } s \le t < \tau \\ [1 - \pi(\tau)k]\,X_1^{\pi,\tau,X_0^\pi(\tau)}(t) & \text{for } t \ge \tau \ge s\,, \end{cases} \quad (8)$$

   given the occurrence of a jump of height k at time τ, is strictly positive. Thereby, it is assumed that the crash time τ is a stopping time, which is supposed to be $\mathcal{F}_t$–measurable. The set of admissible portfolio strategies is obviously given by $A_0(s,x)$ as long as no crash happens. After a crash at time τ the set is given by $A_1(\tau,x)$. Hence,

$$A(s,x) := A_0(s,x)\Big|_{[0,\tau]} \cup A_1(\tau,x).$$

3. *$A(x)$ is used as an abbreviation for $A(0,x)$.*

With these definitions it is possible to state the worst case problem. Note that due to the lack of statistical assumptions on the distribution of both the crash height and the crash time, the problem cannot be dealt with by simply maximizing the expected utility of final wealth. However, the crash consequence has to be taken into account in some way. The approach of this paper is to maximize the worst case possible.

Definition 2.2

1. *Let the utility function U be given by $U(x) = \ln(x)$. Then the problem to solve*
$$\sup_{\pi(\cdot)\in A(x)} \inf_{\substack{0\leq\tau\leq T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi}(T)\right)\right] , \tag{9}$$
where the final wealth $X^{\pi}(T)$ in the case of a crash of size k at time τ is given by
$$X^{\pi}(T) = \left[1 - \pi(\tau)k\right] X_1^{\pi,\tau,X_0^{\pi}(\tau)}(T) , \tag{10}$$
with $X_1^{\pi,\tau,X_0^{\pi}(\tau)}(t)$ as above, is called the **worst case scenario portfolio problem**.

2. *The* **value function** to the above problem *is defined via*
$$\nu_c(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\leq\tau\leq T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right] . \tag{11}$$

3. *The* **value function** in the crash–free setting **of the market model $\boldsymbol{X_i}$** *will be denoted*
$$\nu_i(t,x) = \sup_{\pi(\cdot)\in A_i(t,x)} \mathbb{E}\left[\ln\left(X_i^{\pi,t,x}(T)\right)\right]$$

Clearly, the above defined optimization problems are stochastic control problems. A classical approach to solve a stochastic control problem is to derive the corresponding so–called *Hamilton–Jacobi–Bellman equation*, often abbreviated as *HJB–equation*. For an introduction to this method see e.g. Korn [9]. However, the derivation of the Hamilton–Jacobi–Bellman equation relies on the ***Bellman principle*** (or ***optimality principle***) which asserts that a section of an optimal trajectory is also an optimal trajectory (see Bellman [2], compare also with Korn [9]).

In order to get shorter and more transparent formulae, the following definitions are useful.

Definition 2.3

For $i = 0, 1$ let us name

1. *the* **optimal portfolio strategy in market $\boldsymbol{i}$**, *assuming that no crash will happen, by*
$$\pi_i^* := \frac{\mu_i - r_i}{\sigma_i^2}.$$

2. *Moreover,*
$$\Psi_i := r_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{\sigma_i}\right)^2 = r_i + \frac{\sigma_i^2}{2}\left(\pi_i^*\right)^2$$
will be called the **utility growth potential** *or* **earning potential of market $\boldsymbol{i}$**.

3. *In order to compare the results in this paper with the results of Korn and Wilmott [11] let us name the crash hedging strategy of the market i given that the market conditions do not change after a crash $\hat{\phi}_i$.*

The well–known value function of the crash–free world, given the market coefficients of market i, calculates to

$$\begin{aligned}
\nu_i(t,x) &= \sup_{\pi(\cdot)\in A_i(t,x)} \mathbb{E}\left[\ln\left(X_i^{\pi,t,x}(T)\right)\right] \\
&= \ln(x) + \left(r_i + \frac{1}{2}\left(\frac{\mu_i - r_i}{\sigma_i^2}\right)^2\right)(T-t) \\
&= \ln(x) + \Psi_i\,(T-t)\,.
\end{aligned}$$

In particular, ν_1 is the value–function of the market 1. Hence, ν_1 is the value–function for a crash hedging investor after a crash has happened and no further crash is expected. Moreover, define for an arbitrary admissible portfolio strategy $\pi(t)$

$$\begin{aligned}
\nu_\pi(t,x) &:= \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right] \\
&= \ln(x) + \mathbb{E}\left[\int_t^T \left[\pi(s)\left(\mu_0 - r_0\right) + r_0 - \frac{1}{2}\pi^2(s)\sigma_0^2\right] ds\right] \\
&= \ln(x) - \frac{\sigma_0^2}{2}\mathbb{E}\left[\int_t^T \left[\left(\pi(s) - \pi_0^*\right)^2 - \frac{2}{\sigma_0^2}\Psi_0\right] ds\right] \\
&= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2\right] ds\right].
\end{aligned}$$

This is the utility one gets using the portfolio strategy π in the initial market. Being in the initial market means that no crash has happened so far. If the portfolio strategy is deterministic, the expectation is redundant.

2.2 The Main Results

In order to get the optimal portfolio strategy for an investor, who wants to maximize her worst case scenario portfolio problem, it is easier to calculate the portfolio strategy $\hat{\pi}$ first, which makes the investor crash indifferent. Obviously, the investor is indifferent towards a crash, if her maximized expected worst case final utility before a possible crash is equal to her maximized expected final utility after a crash of the worst possible case. That is, the investor's expected utility is not effected by a crash of the worst possible size. This justifies the following definition, where the convention $\hat{\nu}(t,x) := \nu_{\hat{\pi}}(t,x)$ is used.

Definition 2.4

i) A portfolio strategy $\hat{\pi}$ *determined via the equation*

$$\hat{\nu}\left(t,x\right)=\left\{\begin{array}{ll}\nu_1\left(t,x\left(1-\hat{\pi}(t)k^*\right)\right) & \textit{for } \hat{\pi}(t)\geq 0\\ \nu_1\left(t,x\left(1-\hat{\pi}(t)k_*\right)\right) & \textit{for } \hat{\pi}(t)<0\end{array}\right\}\qquad \textit{for all } t\in[0,T]$$

will be called a **crash hedging strategy**.

ii) A portfolio strategy $\tilde{\pi}$ *is a* **partial crash hedging strategy**, *if there exists an* $S\in(0,T)$ *such that* $\tilde{\pi}$ *is a crash hedging strategy on* $[0,S]$ *and is a solution of the worst case scenario portfolio problem on* $[S,T]$.

Rewriting the determining equation for the non–negative crash hedging strategy $\hat{\pi}$ gives

$$\begin{aligned}
&& \hat{\nu}\left(t,x\right) &= \nu_1\left(t,x\left(1-\hat{\pi}(t)k^*\right)\right)\\
\Longleftrightarrow && \ln\left(x\right)+\int_t^T\left[\Psi_0-\frac{\sigma_0^2}{2}\left(\hat{\pi}(s)-\pi_0^*\right)^2\right]ds &\\
&& &= \ln\left(x\right)+\ln\left(1-\hat{\pi}(t)k^*\right)+\Psi_1\left(T-t\right)\\
\Longleftrightarrow && \ln\left(1-\hat{\pi}(t)k^*\right) &= \int_t^T\left[\Psi_0-\Psi_1-\frac{\sigma_0^2}{2}\left(\hat{\pi}(s)-\pi_0^*\right)^2\right]ds. \qquad (12)
\end{aligned}$$

Differentiating with respect to t yields

$$\begin{aligned}
&& \frac{-\hat{\pi}'(t)k^*}{1-\hat{\pi}(t)k^*} &= \frac{\sigma_0^2}{2}\left(\hat{\pi}(t)-\pi_0^*\right)^2+\Psi_1-\Psi_0\\
\Longleftrightarrow && \hat{\pi}'(t) &= \left(\hat{\pi}(t)-\frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t)-\pi_0^*\right)^2+\Psi_1-\Psi_0\right].
\end{aligned}$$

Clearly, $\hat{\pi}(T)=0$, since the right side of equation (12) is zero for $t=T$ and the left side is only zero for $t=T$, if $\hat{\pi}(T)=0$. Using $\hat{\pi}(T)=0$, this gives

$$\hat{\pi}'(T)\quad=\quad-\frac{1}{k^*}\left(\Psi_1-r_0\right)\quad\left\{\begin{array}{lll}< & 0 & \text{for } \Psi_1>r_0\\ = & 0 & \text{for } \Psi_1=r_0\\ > & 0 & \text{for } \Psi_1<r_0\end{array}\right\}.$$

A close look reveals that $\hat{\pi}'(T)<0$ implies $\hat{\pi}'(t)\leq 0$ for $t\in[0,T)$. Hence, $\hat{\pi}(t)>0$ for $t\in[0,T)$.

Moreover, it is straightforward to verify that $\hat{\pi}'\equiv 0$, if $\hat{\pi}'(T)=0$. Thus, this case yields $\hat{\pi}\equiv 0$. The economic meaning of this being the impossibility to hedge a risky asset if the utility growth potential after a possible crash is only of the size of the initial riskless rate of return.

Finally, the case $\Psi_1 < r_0$ gives

$$\begin{aligned}\nu_\pi(t,x)\Big|_{\pi\equiv 0} &= \ln(x) + r_0(T-t)\\ &> \ln(x) + \Psi_1(T-t)\\ &= \nu_1(t,x) \qquad \text{for } t\in[0,T) \text{ and } x>0.\end{aligned}$$

Thus, the expected worst case is given in this situation by an immediate crash, if the portfolio strategy $\pi \equiv 0$ is used. In order to boost the expected worst case utility, the expected utility after a crash has to be increased. This can only be achieved by going short, i.e. $\pi(t) < 0$ for $t \in [0,T)$. However, if $\hat{\pi}$ is negative, the corresponding differential equation is

$$\hat{\pi}'(t) = \left(\hat{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t)-\pi_0^*\right)^2 + \Psi_1 - \Psi_0\right],$$

which can be confirmed easily. Note that this differential equation has in T the same behavior as the differential equation for non–negative portfolio strategies. This guarantees that the crash hedging strategy is well–defined. Moreover, $\hat{\pi}'(T) > 0$ implies $\hat{\pi}'(t) \geq 0$ for $t \in [0,T)$. Thus, $\hat{\pi}(t) < 0$ for $t \in [0,T)$.

This leads us to the main result of this paper.

Theorem 2.5

1. *If $\Psi_1 \geq r_0$, then there exists a unique crash hedging strategy $\hat{\pi}$, which is given by the solution of the differential equation*

$$\begin{aligned}\hat{\pi}'(t) &= \left(\hat{\pi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t)-\pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], && (13)\\ \text{and}\quad \hat{\pi}(T) &= 0. && (14)\end{aligned}$$

   Moreover, this crash hedging strategy is bounded by $0 \leq \hat{\pi} < \frac{1}{k^*}$. Additionally, if $\Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$, the crash hedging strategy has another upper bound with $\hat{\pi} < \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$.

2. *If $\Psi_1 < r_0$, then there exists a unique crash hedging strategy $\hat{\pi}$, which is given by the solution of the differential equation*

$$\begin{aligned}\hat{\pi}'(t) &= \left(\hat{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t)-\pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], && (15)\\ \text{and}\quad \hat{\pi}(T) &= 0. && (16)\end{aligned}$$

   Furthermore, this crash hedging strategy is bounded by

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} < \hat{\pi}(t) < 0 \qquad \text{for } t \in [0,T).$$

3. *If $\Psi_1 < \Psi_0$ and $\pi_0^* < 0$, there exists a partial crash hedging strategy $\tilde{\pi}$ (which is different from $\hat{\pi}$), if*

$$S := T - \frac{\ln(1 - \pi_0^* k_*)}{\Psi_0 - \Psi_1} > 0. \tag{17}$$

With this, $\tilde{\pi}$ is on $[0, S]$ given by the unique solution of the differential equation

$$\begin{aligned} \tilde{\pi}'(t) &= \left(\tilde{\pi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\tilde{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right], & (18)\\ \text{and} \quad \tilde{\pi}(S) &= \pi_0^*. & (19) \end{aligned}$$

On $[S, T]$ set $\tilde{\pi}(t) := \pi_0^*$. This partial crash hedging strategy is bounded by

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} < \tilde{\pi} \leq \pi_0^* < 0.$$

The optimal portfolio strategy for an investor, who wants to maximize her worst case scenario portfolio problem, is given by

$$\bar{\pi}(t) := \min\left\{\hat{\pi}(t), \tilde{\pi}(t), \pi_0^*\right\} \qquad \text{for all } t \in [0, T], \tag{20}$$

where $\tilde{\pi}(t)$ is only taken into account if it exists. $\bar{\pi}$ will be named the **optimal crash hedging strategy**.

Remark 2.6

1. It is straightforward to verify that $\hat{\pi}$, $\tilde{\pi}$ and $\bar{\pi}$ are admissible portfolio strategies, since they are bounded as well as continuous.

2. Observe that the optimal crash hedging strategy is independent of the crash time τ.

3. Note that the worst case utility bound of the crash hedging strategy is given by $\hat{\nu}(t, x)$, or by $\nu_1\left(t, x\left(1 - \hat{\pi}(t)k^*\right)\right)$, which is according to the construction of $\hat{\pi}$ the same. Hence, it is sufficient to show that either $\nu_\pi(t, x) < \hat{\nu}(t, x)$ or $\nu_1\left(t, x\left(1 - \pi(t)k^*\right)\right) < \nu_1\left(t, x\left(1 - \hat{\pi}(t)k^*\right)\right)$ in order to verify that the portfolio strategy π has a lower expected worst case final utility than $\hat{\pi}$.

4. Compare the differential equation (13) with the differential equation Korn and Wilmott [11] got in Corollary 2.2. Rewriting the above differential equation to

$$\hat{\pi}'(t) = \frac{1}{k^*}\left(1 - \hat{\pi}(t)k^*\right)\left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}\left(\hat{\pi}(t) - \pi_0^*\right)^2\right],$$

it is easy to see that it is up to the correction term $\Psi_1 - \Psi_0$ the same as the differential equation in Korn and Wilmott [11].

5. Notice that the investor is only indifferent between no crash and a crash of the worst case possible. In general, any other crash of size k with $k_* < k < k^*$ will be favorable for the investor, if the investor uses the crash hedging strategy $\hat{\pi}$.

6. Observe that the investor will P–a.s. not go bankrupt, if he uses the portfolio strategy $\hat{\pi}$. For example in the case of $\pi_0^* \geq 0$, $\hat{\pi} < 0$ and $\pi_B \equiv 0$ (the pure bond strategy):

$$\hat{\nu}_{\pi_B}(t,x) > \hat{\nu}(t,x) = \nu_1\left(t, x\left(1 - \hat{\pi}(t)k_*\right)\right) > \nu_1(t,x).$$

Since the last case is the classical utility function in the crash–free model where the investor do not go bankrupt, the investor can not go bankrupt in the other cases as well.

The following lemmata will prepare the proof of Theorem 2.5 and will reveal some important properties of the crash hedging strategy $\hat{\pi}$.

Lemma 2.7
Any admissible portfolio strategy π which satisfies

$$\mathbb{E}\left[\pi(t)\right] < \hat{\pi}(t) \leq \pi_0^* \qquad \textit{for all } t \in [0,T]$$

has a lower expected worst case utility bound than $\hat{\pi}$, the crash hedging strategy.

Proof: Using the Theorem of Fubini and the the fact that

$$\mathbb{E}\left[X^2\right] = \left(\mathbb{E}\left[X\right]\right)^2 + \mathrm{Var}(X) \geq \left(\mathbb{E}\left[X\right]\right)^2$$

for any square integrable random variable X, the case of no crash occurring gives

$$\begin{aligned}
\nu_\pi(t,x) &= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2\right] ds\right] \\
&\leq \ln(x) + \int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\mathbb{E}\left[\pi(s)\right] - \pi_0^*\right)^2\right] ds \\
&< \ln(x) + \int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\hat{\pi}(s) - \pi_0^*\right)^2\right] ds \\
&= \hat{\nu}\left(t,x\right),
\end{aligned}$$

which shows that $\pi(t)$ has a lower expected worst case utility bound than $\hat{\pi}$. □

Lemma 2.8
Of all admissible portfolio strategies π with

$$\mathbb{E}\left[\pi(t)\right] < \hat{\pi}(t) \qquad \textit{for all } t \in I := \{t : \hat{\pi}(t) > \pi_0^*\} \tag{21}$$

$\bar{\pi}$ yields the highest worst case utility bound (where $\bar{\pi}$ is defined in (20) in Theorem 2.5).

Remark 2.9
Observe that I is a disjoint union of closed intervals, since $\hat{\pi}$ is continuously differentiable. Actually, later on the following can be verified. The case $I \neq \emptyset$ can only happen if $\Psi_1 - \Psi_0 > 0$. However, in this case $\hat{\pi}$ is strictly decreasing, thus I is an interval of the form $I = [0, t_0]$.

Proof: For any admissible portfolio strategy π with

$$\mathbb{E}\left[\pi(t)\right] < 2\pi_0^* - \hat{\pi}(t) \qquad \text{for all } t \in I$$

Lemma 2.7 applies basically analog. Hence, let us restrict to portfolio strategies π, which satisfy

$$2\pi_0^* - \hat{\pi}(t) < \mathbb{E}\left[\pi(t)\right] < \hat{\pi}(t) \qquad \text{for some } t \in I.$$

Without loss of generality, let us assume that

$$2\pi_0^* - \hat{\pi}(t) < \mathbb{E}\left[\pi(t)\right] < \hat{\pi}(t) \qquad \text{for all } t \in I.$$

For simplicity, let us suppose that I is of the form $I = [t_0, t_1]$. Choosing $t \in I$, the above inequality implies

$$\begin{aligned}
&\ln(x) + \mathbb{E}\left[\int_t^{t_1} \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2\right] ds\right] \\
&= \ln(x) + \int_t^{t_1} \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\mathbb{E}\left[\pi(s)\right] - \pi_0^*\right)^2 - \frac{\sigma_0^2}{2}\mathrm{Var}\left(\pi(s)\right)\right] ds \\
&> \ln(x) + \int_t^{t_1} \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\hat{\pi}(s) - \pi_0^*\right)^2 - \frac{\sigma_0^2}{2}\mathrm{Var}\left(\pi(s)\right)\right] ds.
\end{aligned} \tag{22}$$

The last inequality shows that any portfolio strategy $\pi(t)$ satisfying (21) and with a variance small enough has a higher expected final utility than the crash hedging strategy $\hat{\pi}(t)$, if no crash occurs and if $t \in I$. It is straightforward to verify that of these strategies only π_0^* maximizes the expected final utility in the interval I, if no crash happens.

Lemma 2.7 justifies that – without loss of generality – it is possible to set $\pi(t) = \hat{\pi}(t)$ for $t \notin I$. Using this together with the above inequality (22) gives

$$\nu_\pi(t,x) > \hat{\nu}(t,x) - \frac{\sigma_0^2}{2} \int_t^{t_1} \mathrm{Var}\left(\pi(s)\right) ds,$$

which shows that any portfolio strategy π satisfying (21) and with a variance small enough has a higher expected final utility than the crash hedging strategy $\hat{\pi}$, if no crash occurs. It is straightforward to verify that of these strategies only $\bar{\pi}$ maximizes the expected final utility, if no crash happens. In order to show that this maximizes the expected worst case utility bound, one has to consider the following two cases.

i) $\pi_0^* \geq 0$:
It suffices to verify that for all $t \in I$

$$\begin{aligned} & \nu_{\bar{\pi}}(t,x) && \leq \nu_1(t, x(1-\pi_0^* k^*)) \\ \iff \quad & \int_t^T \left[\Psi_0 - \Psi_1 - \frac{\sigma^2}{2}\left(\bar{\pi}(s) - \pi_0^*\right)^2\right] ds && \leq \ln\left(1 - \pi_0^* k^*\right). \end{aligned}$$

Since $\hat{\pi}$ is decreasing and ending at $\hat{\pi}(T) = 0$, there exists an $S \in [0,T)$ such that $\bar{\pi}(s) = \hat{\pi}(s)$ for $s \in [S,T]$. The important case which has to be considered is $S \in (t,T)$. Hence, the above reduces to

$$\begin{aligned} & \left(\Psi_0 - \Psi_1\right)(S-t) + \underbrace{\int_S^T \left[\Psi_0 - \Psi_1 - \frac{\sigma^2}{2}\left(\hat{\pi}(s) - \pi_0^*\right)^2\right] ds}_{=\ln\left(1-\pi_0^* k^*\right)} && \leq \ln\left(1 - \pi_0^* k^*\right) \\ \iff \quad & \Psi_1 && \geq \Psi_0. \end{aligned}$$

This last inequality can be verified easily, since it is only possible that $\hat{\pi} \geq \pi_0^*$, if $\Psi_1 \geq \Psi_0$.

ii) $\pi_0^* < 0$:
If $\Psi_1 \geq \Psi_0$ than it is straightforward to verify that

$$\nu_\pi(t,x) \leq \nu_1(t, x(1 - \pi(t)k_*))$$

for any admissible portfolio strategy π with $\pi \leq 0$. Thus, let us consider the case $\Psi_1 < \Psi_0$. It remains to confirm that

$$\nu_{\bar{\pi}}(t,x) \leq \nu_1(t, x(1 - \bar{\pi}(t)k_*))$$

$$\Longleftrightarrow \qquad \int_t^T \left[\Psi_0 - \Psi_1 - \frac{\sigma^2}{2} \left(\bar{\pi}(s) - \pi_0^* \right)^2 \right] ds \quad \leq \quad \ln\left(1 - \bar{\pi}(t) k_*\right).$$

Assume that $\bar{\pi} \equiv \pi_0^*$. This leads to

$$\begin{aligned} (\Psi_0 - \Psi_1)(T-t) &\leq \ln(1 - \bar{\pi}(t)k_*) \\ \Longleftrightarrow \qquad t &\geq T - \frac{\ln(1 - \bar{\pi}(t)k_*)}{\Psi_0 - \Psi_1} \end{aligned}$$

The right side of the last inequality has been defined in (17) to be S. This inequality shows that π_0^* is an optimal worst case portfolio strategy on $[\max(S, 0), T]$. Obviously, at time S equality holds. Suppose that $S > 0$, then

$$\nu_{\pi_0^*}(t, x) > \nu_1(t, x(1 - \pi_0^* k_*)) \qquad \text{for } t \in [0, S).$$

In this situation the worst case is given by an immediate crash. However, it is clearly possible to improve this worst case utility by reducing π_0^*. Equality is by construction reached for $\tilde{\pi}$. Since ν_π is strictly increasing for $\pi < \pi_0^*$ and $\nu_1(t, x(1 - \pi k_*))$ is strictly decreasing for π, it is straightforward to verify that $\tilde{\pi}$ is optimal in this situation.

This concludes the assertion. □

Lemma 2.10
Any admissible portfolio strategy π which satisfies

$$\mathbb{E}[\pi(t)] > \bar{\pi}(t) \qquad \textit{for some } t \in [0, T] \tag{23}$$

has a lower worst case utility bound than the optimal crash hedging strategy $\bar{\pi}$.

Proof: First, let us suppose that $\mathbb{E}[\pi(0)] \leq \bar{\pi}(0)$ as well as $\mathbb{E}[\pi(T)] \leq \bar{\pi}(T)$. Hence, there exist $t^\star \in [0, T)$ and $\varepsilon > 0$ such that

$$\mathbb{E}[\pi(t)] \leq \bar{\pi}(t) \quad \text{for } t \in [0, t^\star] \qquad \text{and} \qquad \mathbb{E}[\pi(t)] > \bar{\pi}(t) \quad \text{for } t \in (t^\star, \varepsilon].$$

Such a construction is always possible due to assumption (1ii) in Definition 2.1. Without loss of generality, let us suppose that $\pi(t) = \bar{\pi}(t)$ for $t \in [0, t^\star]$.

It suffices to show that

$$\nu_1(t, x(1 - \pi(t)k^*)) < \nu_1(t, x(1 - \bar{\pi}(t)k^*)) \qquad \text{for } t \in (t^\star, \varepsilon].$$

However, this can be seen straightforward

$$\begin{aligned} \nu_1(t, x(1 - \pi(t)k^*)) &= \ln(x) + \mathbb{E}[\ln(1 - \pi(t)k^*)] + \Psi_1(T-t) \\ &\leq \ln(x) + \ln(1 - \mathbb{E}[\pi(t)]k^*) + \Psi_1(T-t) \end{aligned}$$

$$\begin{aligned} &< \ln(x) + \ln\left(1 - \bar{\pi}(t)k^*\right) + \Psi_1(T-t) \\ &= \nu_1\left(t, x\left(1 - \bar{\pi}(t)k^*\right)\right). \end{aligned} \tag{24}$$

Hence, an immediate crash of the worst possible size at time t gives a lower final expected utility for the portfolio strategy π than for the crash hedging strategy $\hat{\pi}$.

It is easy to verify that inequality (24) holds for $t = 0$ or $t = T$ as well, if $\mathbb{E}\left[\pi(0)\right] > \bar{\pi}(0)$ or $\mathbb{E}\left[\pi(T)\right] > \bar{\pi}(T)$, respectively.

Since $\nu_1\left(t, x\left(1 - \bar{\pi}(t)k^*\right)\right)$ is already the worst case utility bound for $\bar{\pi}$, this proves the assertion. $\square$

Proof of Theorem 2.5: The differential equations have been derived above. Furthermore, it is straightforward to verify that any crash hedging strategy has to satisfy $\hat{\pi}(T) = 0$.

$$\begin{aligned} \hat{\nu}\left(T, x\right) &= \nu_1\left(T, x\left(1 - \hat{\pi}(T)k^*\right)\right) \\ \Longleftrightarrow \qquad \ln\left(x\right) &= \ln\left(x\right) + \ln\left(1 - \hat{\pi}(T)k^*\right) \end{aligned}$$

Since $k^* \neq 0$, this is only possible for $\hat{\pi}(T) = 0$.

Let us prove that $\hat{\pi}$ is bounded. The upper bound $\frac{1}{k^*}$ of $\hat{\pi}$ can be verified in equation (12) due to the fact that this equation has a pole for $\hat{\pi}(t) = \frac{1}{k^*}$ for an arbitrary $t \in [0, T]$. The economic meaning of this being that the investor would be bankrupt in case of $\hat{\pi}(t) \geq \frac{1}{k^*}$ and a crash of size k^* at time t. The bound $\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$ is due to the fact that it is a zero of the right side of (13) or (15), if $\Psi_1 \leq \Psi_0$. This bound will never be reached by the continuous crash hedging strategy, since this would imply that the continuous crash hedging strategy would eventually become constant, which can not be the case. The other bounds have been shown above.

In order to prove the existence and uniqueness of a solution for the differential equation let us denote

$$F\left(t, y\right) := \left(y - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(y - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right].$$

Clearly, F is a polynomial in y which is independent of t. Hence, F is continuously partial differentiable and therefore especially locally Lipschitz–continuous with respect to y (see e.g. Forster [6], Satz 1, p. 102). This gives then already the uniqueness of the solution for the differential equation (13) with terminal value (14) (see e.g. Forster [6], Satz 2, p. 102).

Moreover, since F is continuous and locally Lipschitz–continuous on $[0, T] \times \mathbb{R}$ the theorem of Picard–Lindelöf gives the existence of a solution for the differential equation (13) with terminal value (14) on a suitable neighbourhood of any arbitrary point $y \in \mathbb{R}$ (see e.g. Heuser [7], Satz 12.4, p. 141). This gives the existence of a solution for the differential equation on any compact set $[a,b] \subset \mathbb{R}$ with $a < b$. To see this define $F_{[a,b]}$ to be equivalent to F on $[a,b]$, to have compact support, and to be continuously differentiable. The construction of such a function is well–known. Obviously, $F_{[a,b]}$ is continuous and global Lipschitz–continuous. Thus proving the existence of a solution for the differential equation which corresponds to $F_{[a,b]}$ (see e.g. Heuser [7]). Choosing a and b sufficiently large, this proves the existence of a solution for the differential equation (13), since $\hat{\pi}$ is bounded.

Replacing k^* by k_* shows that the same statement holds for the differential equation (15) with terminal value (16).

Applying Lemma 2.7, Lemma 2.8 and Lemma 2.10 gives the optimality of $\bar{\pi}$ in both cases. All in all, this concludes the Theorem. □

In order to solve the differential equation (13), note that this differential equation can be rewritten as

$$\begin{aligned} \int_{t_0}^{t_1} \frac{d\hat{\pi}(t)}{\left(\hat{\pi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right]} &= \int_{t_0}^{t_1} dt \\ \Longleftrightarrow \quad \int_{t_0}^{t_1} \frac{d\hat{\pi}(t)}{\left(\frac{1}{k^*} - \hat{\pi}(t)\right)\left[\left(\hat{\pi}(t) - \pi_0^*\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)\right]} &= -\frac{\sigma_0^2}{2}\int_{t_0}^{t_1} dt, \end{aligned} \tag{25}$$

with $t_0, t_1 \in [0,T]$. Calculating the zeros of the polynomial for the partial fraction expansion delivers

$$(a - \pi_0^*)^2 = \frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right) \quad \Longleftrightarrow \quad a_{1/2} = \pi_0^* \pm \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}.$$

There are three general different cases to consider.

i) $\Psi_1 = \Psi_0$ and $\frac{1}{k^*} \neq \pi_0^* \implies a = \pi_0^*$ is a double zero.

ii) $\Psi_1 < \Psi_0$ and $\frac{1}{k^*} \neq \pi_0^* \pm \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} \implies a_{1/2} = \pi_0^* \pm \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$ are two different real zeros.

iii) $\Psi_1 > \Psi_0 \implies a_{1/2} = \pi_0^* \pm \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$ are complex zeros.

Furthermore, there are three special cases to be taken into consideration.

iv) $\Psi_1 = \Psi_0$ and $\frac{1}{k^*} = \pi_0^* \implies a = \pi_0^*$ is a triple zero.

v) $\Psi_1 < \Psi_0$ and $\frac{1}{k^*} = \pi_0^* + \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} \implies a_1 = \pi_0^* + \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$ is a double zero and $a_2 = \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$ is a single zero.

vi) $\Psi_1 < \Psi_0$ and $\frac{1}{k^*} = \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}$ $\Longrightarrow$ $a_1 = \pi_0^* + \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}$ is a single zero and $a_2 = \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}$ is a double zero.

Applying the partial fraction expansion in each of the above six cases will give a solution for each case. In order to enable a concise representation of the characterization for the solutions define

$$\Theta_{\pm} := \left(\pi_0^* - \frac{1}{k^*} \pm \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}\right), \tag{26}$$

$$\Delta := \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}, \tag{27}$$

and

$$\Delta_1 := \sqrt{\frac{2}{\sigma_0^2}(\Psi_1 - \Psi_0)}. \tag{28}$$

Proposition 2.11

With these conventions one has the following characterizations for the solutions of the differential equation (13) with the terminal condition (14).

i) If $\Psi_1 = \Psi_0$ and $\frac{1}{k^*} \neq \pi_0^*$, then

$$\frac{\sigma_0^2}{2}\left(\pi_0^* - \frac{1}{k^*}\right)^2 (T-t) = \ln\left(\frac{1 - \frac{\hat{\pi}(t)}{\pi_0^*}}{1 - \hat{\pi}(t)k^*}\right) + \left(1 - \frac{1}{k^*\pi_0^*}\right)\frac{\hat{\pi}(t)}{\hat{\pi}(t) - \pi_0^*}.$$

ii) If $r_0 \leq \Psi_1 < \Psi_0$ and $\frac{1}{k^*} \neq \pi_0^* \pm \Delta$, then

$$\begin{aligned} \Theta_+ \cdot \Theta_- \cdot \Delta \cdot \sigma_0^2 (T-t) &= -2\cdot\Delta\cdot\ln(1 - \hat{\pi}(t)k^*) - \Theta_- \cdot \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* + \Delta}\right) \\ &\quad + \Theta_+ \cdot \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* - \Delta}\right). \end{aligned}$$

iii) If $\Psi_1 > \Psi_0$, then

$$\begin{aligned} \Delta_1\left[\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \Delta_1^2\right]\sigma_0^2(T-t) &= -2\cdot\Delta_1\cdot\ln(1 - \hat{\pi}(t)k^*) \\ &\quad + \Delta_1 \cdot \ln\left(\frac{(\hat{\pi}(t) - \pi_0^*)^2 + \Delta_1^2}{(\pi_0^*)^2 + \Delta_1^2}\right) \\ &\quad - 2\left(\pi_0^* - \frac{1}{k^*}\right)\arctan\left(\frac{\Delta_1 \cdot \hat{\pi}(t)}{\Delta_1^2 - \hat{\pi}(t)\pi_0^* + (\pi_0^*)^2}\right). \end{aligned} \tag{29}$$

iv) If $\Psi_1 = \Psi_0$ *and* $\frac{1}{k^*} = \pi_0^*$*, then*

$$\hat{\pi}(t) = \pi_0^* - \frac{\pi_0^*}{\sqrt{(\pi_0^*)^2 \sigma_0^2 (T-t) + 1}}.$$

v) If $r_0 \le \Psi_1 < \Psi_0$ *and* $\frac{1}{k^*} = \pi_0^* + \Delta$*, then*

$$2\Delta^2 \cdot \sigma_0^2 (T-t) = \ln(1 - \hat{\pi}(t)k^*) - \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* - \Delta}\right) - 2\Delta \cdot \frac{\hat{\pi}(t)(k^*)^2}{1 - \hat{\pi}(t)k^*}.$$

vi) If $r_0 \le \Psi_1 < \Psi_0$ *and* $\frac{1}{k^*} = \pi_0^* - \Delta$*, then*

$$2\Delta^2 \cdot \sigma_0^2 (T-t) = \ln(1 - \hat{\pi}(t)k^*) - \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* + \Delta}\right) + 2\Delta \cdot \frac{\hat{\pi}(t)(k^*)^2}{1 - \hat{\pi}(t)k^*}.$$

Proof: See appendix. □

Notice that all cases – except the fourth – have only implicit solutions. However, it is straightforward to get in each case the inverse function of $\hat{\pi}(t)$.

With the above implicit solutions it is obvious what the characterizations for the solutions of the differential equation (15) are. However, there are only three different cases to consider (since the other three cases are not possible). Denote

$$\Theta_\pm^* := \left(\pi_0^* - \frac{1}{k_*} \pm \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}\right).$$

Corollary 2.12
The characterizations for the solutions of the differential equation (15) with the terminal condition (16) can be written as follows.

i) If $\Psi_1 < r_0$ *and* $\frac{1}{k_*} \ne \pi_0^* \pm \Delta$*, then*

$$\begin{aligned} \Theta_+^* \cdot \Theta_-^* \cdot \Delta \cdot \sigma_0^2 (T-t) &= -2 \cdot \Delta \cdot \ln(1 - \hat{\pi}(t)k_*) - \Theta_-^* \cdot \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* + \Delta}\right) \\ &\quad + \Theta_+^* \cdot \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* - \Delta}\right). \end{aligned}$$

ii) If $\Psi_1 < r_0$ *and* $\frac{1}{k_*} = \pi_0^* + \Delta$*, then*

$$2\Delta^2 \cdot \sigma_0^2 (T-t) = \ln(1 - \hat{\pi}(t)k_*) - \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* - \Delta}\right) - 2\Delta \cdot \frac{\hat{\pi}(t)(k_*)^2}{1 - \hat{\pi}(t)k_*}.$$

iii) If $\Psi_1 < r_0$ *and* $\frac{1}{k_*} = \pi_0^* - \Delta$*, then*

$$2\Delta^2 \cdot \sigma_0^2 (T-t) \;=\; \ln(1-\hat{\pi}(t)k_*) - \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^* + \Delta}\right) + 2\Delta \cdot \frac{\hat{\pi}(t)\,(k_*)^2}{1-\hat{\pi}(t)k_*}.$$

Corollary 2.13
The characterizations for the solutions of the differential equation (18) with the terminal condition (19) are the following.

i) If $\Psi_1 < \Psi_0$ *and* $\frac{1}{k_*} \neq \pi_0^* + \Delta$*, then*

$$\begin{aligned}\Theta_+^* \cdot \Theta_-^* \cdot \Delta \cdot \sigma_0^2 (S-t) \;&=\; 2\Delta \cdot \ln\left(\frac{1-\pi_0^* k_*}{1-\tilde{\pi}(t)k_*}\right) + \Theta_-^* \cdot \ln\left(\frac{\Delta}{\pi_0^* + \Delta - \tilde{\pi}(t)}\right) \\ &\quad - \Theta_+^* \cdot \ln\left(\frac{\Delta}{\tilde{\pi}(t) - \pi_0^* + \Delta}\right).\end{aligned}$$

ii) If $\Psi_1 < r_0$ *and* $\frac{1}{k_*} = \pi_0^* + \Delta$*, then*

$$2\Delta^2 \cdot \sigma_0^2 (S-t) \;=\; \ln\left(\frac{\Delta}{\tilde{\pi}(t) - \pi_0^* + \Delta}\right) - \ln\left(\frac{1-\pi_0^* k_*}{1-\tilde{\pi}(t)k_*}\right) - 2 \cdot \frac{\tilde{\pi}(t) - \pi_0^*}{\frac{1}{k_*} - \tilde{\pi}(t)}.$$

Proof: The proof is similar to the proof of Corollary 2.12. Only, keep in mind that $\tilde{\pi}(S) = \pi_0^*$. Moreover, since in this case $\pi_0^* < 0$ always holds, case iii) can not happen and case ii) holds only for $\Psi_1 < r_0$ (otherwise $k_* < 0$ which is not allowed). □

2.3 Inflection Points of the Crash Hedging Strategy

As it can be seen in the figures (see Figure 4), the crash hedging strategy can have up to two inflection points. To determine the inflection points, it is necessary to calculate the second derivative of $\hat{\pi}$.

$$\begin{aligned}\hat{\pi}''(t) \;&=\; \left[\frac{\sigma_0^2}{2}(\hat{\pi}(t) - \pi_0^*)^2 + \Psi_1 - \Psi_0\right] + \left(\hat{\pi}(t) - \frac{1}{k^*}\right)\sigma_0^2 (\hat{\pi}(t) - \pi_0^*) \\ &=\; \frac{3\sigma_0^2}{2}\hat{\pi}^2(t) - \sigma_0^2\left(2\pi_0^* + \frac{1}{k^*}\right)\hat{\pi}(t) + \frac{\sigma_0^2}{2}(\pi_0^*)^2 + \sigma_0^2\frac{\pi_0^*}{k^*} + \Psi_1 - \Psi_0 \\ &=\; \frac{3\sigma_0^2}{2}\left[\left(\hat{\pi}(t) - \frac{1}{3}\left(2\pi_0^* + \frac{1}{k^*}\right)\right)^2 - \frac{1}{9}\left(2\pi_0^* + \frac{1}{k^*}\right)^2\right. \\ &\qquad \left. + \frac{1}{3}(\pi_0^*)^2 + \frac{2}{3}\frac{\pi_0^*}{k^*} + \frac{2}{3\sigma_0^2}(\Psi_1 - \Psi_0)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{3\sigma_0^2}{2}\left[\left(\hat{\pi}(t) - \frac{1}{3}\left(2\pi_0^* + \frac{1}{k^*}\right)\right)^2\right.\\
&\qquad \left. - \frac{1}{9}\left(\pi_0^*\right)^2 + \frac{2}{9}\frac{\pi_0^*}{k^*} - \frac{1}{9}\frac{1}{\left(k^*\right)^2} + \frac{2}{3\sigma_0^2}\left(\Psi_1 - \Psi_0\right)\right]\\
&= \frac{3\sigma_0^2}{2}\left[\left(\hat{\pi}(t) - \frac{1}{3}\left(2\pi_0^* + \frac{1}{k^*}\right)\right)^2\right.\\
&\qquad \left. - \frac{1}{9}\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{2}{3\sigma_0^2}\left(\Psi_1 - \Psi_0\right)\right].
\end{aligned}
$$

Setting $\hat{\pi}''(t) = 0$ gives the potential candidates for the inflection points.

$$
z_{1/2} = \frac{1}{3}\left(2\pi_0^* + \frac{1}{k^*}\right) \pm \sqrt{\frac{1}{9}\left(\pi_0^* - \frac{1}{k^*}\right)^2 - \frac{2}{3\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}.
$$

These points are only relevant if they satisfy $0 \leq z_{1/2} < \frac{1}{k^*}$. It remains to check the sufficient and necessary condition of an inflection point. Therefore, let us specify the third derivative of $\hat{\pi}(t)$.

$$
\hat{\pi}'''(t) = 3\sigma_0^2\left(\hat{\pi}(t) - \frac{1}{3}\left(2\pi_0^* + \frac{1}{k^*}\right)\right).
$$

Thus,

$$
\hat{\pi}'''(t)\Bigg|_{z_{1/2}} = \pm\, 3\sigma_0^2\sqrt{\frac{1}{9}\left(\pi_0^* - \frac{1}{k^*}\right)^2 - \frac{2}{3\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}.
$$

Since

$$
\hat{\pi}'''(t)\Bigg|_{z_{1/2}} = 0 \qquad \Longleftrightarrow \qquad \left(\pi_0^* - \frac{1}{k^*}\right)^2 = \frac{6}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right),
$$

and since all derivatives of order 5 or higher are equal to zero, $z_{1/2}$ are inflection points if

$$
\left(\pi_0^* - \frac{1}{k^*}\right)^2 \neq \frac{6}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right).
$$

2.4 The Constant Crash Hedging Strategy and the Best Worst Case Constant Portfolio Strategy

Let us consider in this section only constant portfolio strategies. In this case, there are only three different potential scenarios for the worst case strategy to consider. First, a crash at the beginning, second a crash at the end, and third no crash. This is since neither the market coefficient nor the portfolio strategy changes over time. Suppose for a start that $\mu_0 \geq r_0$, hence implying $\pi_0^* \geq 0$. The defining equation for the constant crash hedging strategy $\hat{\varphi}$ is given by the utility indifference equation (see Definition 2.4). Calculating *constant* crash hedging strategies, however, it might no longer be possible that the investor is crash–indifferent (see also Remark 2.14, point 3. below). Keeping this in mind, the wording utility indifference equation will still be used.

Assuming that $\varphi \geq 0$ and φ is a constant portfolio stategy, the worst case is either a crash at the beginning or a crash at the end of the investment period. Thus, the utility indifference equation writes as

$$\begin{aligned}
& & \bar{\nu}_{\varphi}(t,x) &= \bar{\nu}_1\left(t, x\left(1-\varphi k^*\right)\right) \\
&\Longleftrightarrow & \ln(x) + \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\varphi - \pi_0^*\right)^2\right](T-t) + \ln\left(1-\varphi k^*\right) & \\
& & &= \ln(x) + \ln\left(1-\varphi k^*\right) + \Psi_1 (T-t) \\
&\Longleftrightarrow & \left(\varphi - \pi_0^*\right)^2 &= \frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right) \\
&\Longleftrightarrow & \varphi_{1/2} &= \pi_0^* \pm \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}. \qquad (30)
\end{aligned}$$

Obviously, the above equation holds only for $\Psi_0 \geq \Psi_1 \geq r_0$. For $\Psi_0 < \Psi_1$ there exists no constant crash hedging strategy, since the utility growth potential after a possible crash is greater than the utility growth potential in the initial market. Although φ_1 and φ_2 are both crash hedging strategies, clearly $\bar{\nu}_1\left(t, x\left(1-\varphi_1 k^*\right)\right) < \bar{\nu}_1\left(t, x\left(1-\varphi_2 k^*\right)\right)$. Hence, any rational crash hedging investor will choose the constant portfolio strategy φ_2. However, one has $\varphi_2 < 0$ for $\Psi_1 < r_0$. In the case of $\varphi < 0$, the worst case is given by either a crash at the beginning or no crash. Thus, the utility indifference equation is given by no crash at the end.

$$\begin{aligned}
& & \bar{\nu}_{\varphi}(t,x) &= \bar{\nu}_1\left(t, x\left(1-\varphi k_*\right)\right) \\
&\Longleftrightarrow & \ln(x) + \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\varphi - \pi_0^*\right)^2\right](T-t) & \\
& & &= \ln(x) + \ln\left(1-\varphi k_*\right) + \Psi_1 (T-t) \\
&\Longleftrightarrow & \ln\left(1-\varphi k_*\right) &= \left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}\left(\varphi - \pi_0^*\right)^2\right](T-t)
\end{aligned}$$

Given that $t < T$, the right side can only be positive for $\varphi < 0$ if $\Psi_1 < r_0$. Moreover, the left side ranges from 0 to infinity for φ going from 0 to minus infinity. The right side ranges from $r_0 - \Psi_1 > 0$ to minus infinity for φ going from 0 to minus infinity. According to the mean value theorem, there exists a φ such that equality holds, since both sides are continuous on $[0, -\infty)$. Moreover, since both sides are strictly concave functions in φ, there is only one unique intersection point.

Denoting this solution by $\varphi_{1,t}$, this is the constant crash hedging strategy on the time interval $[t, T]$, if $\Psi_1 < r_0$. Hence, the constant crash hedging strategy on the time interval $[t, T]$ is given by

$$\hat{\varphi}_t := \left\{ \begin{array}{cl} \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}, & \text{for } r_0 \leq \Psi_1 \leq \Psi_0 \\ \varphi_{1,t}, & \text{for } \Psi_1 < r_0 \end{array} \right\}.$$

Remark 2.14

1. Note that $\hat{\varphi} = \pi_0^*$, if $\Psi_0 = \Psi_1$. This means that in this peculiar case the constant crash hedging strategy is given by the classical optimal portfolio strategy.

2. Observe that $\hat{\varphi} < 0$, if $\Psi_1 < r_0$ as in the case of non-constant portfolio strategies. Note further that in this case as in the case of $\hat{\varphi} \leq \pi_0^* < 0$ the investor transfers utility from the initial market into the market after a possible crash.

3. If the constant crash hedging strategy is independent of the investment period $T - t$ (as it is the case for $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$), then this constant crash hedging strategy makes the investor crash indifferent for any time s, with $t \leq s \leq T$:

$$\begin{aligned} \ln(x) + \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\hat{\varphi} - \pi_0^*\right)^2\right](s-t) + \ln\left(1 - \hat{\varphi}k^*\right) + \Psi_1\left(T - s\right) \\ = \ln(x) + \Psi_1\left(s - t\right) + \ln\left(1 - \hat{\varphi}k^*\right) + \Psi_1\left(T - s\right) \\ = \bar{\nu}_1\left(t, x\left(1 - \bar{\varphi}k^*\right)\right). \end{aligned}$$

   On the other hand, if the constant crash hedging strategy depends on the investment period $T - t$ (as it is the case for $\Psi_1 < r_0$ and $\pi_0^* \geq 0$), the investor is only crash independent at the times t and T. Hence, these constant crash hedging strategies are only **weak crash hedging strategies**.

As it has been mentioned, the constant crash hedging strategy is independent of the investment time horizon $T - t$, if $r_0 \leq \Psi_1 \leq \Psi_0$. This already indicates that it is not the best constant portfolio strategy. For example, if the investment horizon is very small, $\pi_B \equiv 0$ (the so–called pure bond strategy) will lead to a higher expected final utility.

In order to determine the best constant worst case portfolio strategy, keep in mind that $\bar{\nu}_1\left(t, x\left(1 - \hat{\varphi}k^*\right)\right)$ is monotonously decreasing in φ for $\varphi < 1/k^*$. Thus, the best constant portfolio strategy in the worst case scenario is less or equal than $\hat{\varphi}$. Therefore, it suffices to determine the smallest maximum of $\bar{\nu}_\varphi(t, x)$, which is less or equal to $\hat{\varphi}$.

Before calculating the smallest maximum of $\bar{\nu}_\varphi(t,x)$, let us consider the case $\Psi_1 > \Psi_0$. In this case a crash would be favorable for the investor, since the earning potential after a possible crash is at least as good as before a crash. And after a crash – with no other crash fearing – π_1^*, the classical optimal portfolio strategy would be optimal. Thus, the worst would be a crash at the end of the investment period, since the investor can not take advantage of the better market conditions. Hence, the worst is a crash of size k^* at time T. Using this fact, the expected worst case utility of the investor calculates to

$$\begin{aligned} v(\varphi) &:= \bar{\nu}_\varphi(t,x) \\ &= \ln(x) + \left(\Psi_0 - \frac{\sigma_0^2}{2}\left(\varphi - \pi_0^*\right)^2\right)(T-t) + \ln\left(1-\varphi k^*\right), \end{aligned}$$

if $\varphi \geq 0$ is supposed. Taking the first and second derivative yields

$$\begin{aligned} v'(\varphi) &= -\sigma_0^2\left(\varphi - \pi_0^*\right)(T-t) + \frac{1}{\varphi - \frac{1}{k^*}} \\ v''(\varphi) &= -\sigma_0^2(T-t) - \frac{1}{\left(\varphi - \frac{1}{k^*}\right)^2}. \end{aligned}$$

Notice that the second derivative is negative for $t \neq T$ or for finite φ with $\varphi \neq 1/k^*$. Setting the first derivative equal to zero gives the potential candidates for the best constant worst case portfolio strategy.

$$\begin{aligned} & \left(\varphi - \pi_0^*\right)\left(\varphi - \frac{1}{k^*}\right) &&= \frac{1}{\sigma_0^2(T-t)} \\ \Longleftrightarrow\quad & \varphi^2 - \left(\pi_0^* + \frac{1}{k^*}\right)\varphi + \frac{\pi_0^*}{k^*} &&= \frac{1}{\sigma_0^2(T-t)} \\ \Longleftrightarrow\quad & \left(\varphi - \frac{1}{2}\left(\pi_0^* + \frac{1}{k^*}\right)\right)^2 &&= \frac{1}{4}\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{1}{\sigma_0^2(T-t)} \\ \Longleftrightarrow\quad & \varphi_{1/2} &&= \frac{1}{2}\left(\pi_0^* + \frac{1}{k^*}\right) \pm \sqrt{\frac{1}{4}\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{1}{\sigma_0^2(T-t)}} \end{aligned}$$

Both points are easily verified to be maxima. However, $1-\varphi_1 k^* \leq 0$ which means that the investor would go bankrupt if he uses the portfolio strategy φ_1 and a crash of size k^* happens. This fact leaves only φ_2 as a potential best constant worst case portfolio strategy. Note that $\varphi_2 > 0$, if $\pi_0^* > \frac{k^*}{\sigma_0^2(T-t)}$.

Proposition 2.15
Let be $\mu_0 \geq r_0$. Moreover, denote the constant crash hedging strategy on the time interval $[t,T]$ by $\hat{\varphi}_t$ and correspondingly the best constant worst case portfolio strategy on the time interval $[t,T]$ by $\bar{\varphi}_t$.

1. *For $\Psi_1 > \Psi_0$, there exist no constant crash hedging strategy. However, the best constant worst case portfolio strategy is*

$$\bar{\varphi}_t = \left[\frac{1}{2}\left(\pi_0^* + \frac{1}{k^*}\right) - \sqrt{\frac{1}{4}\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{1}{\sigma_0^2\,(T-t)}}\right]^+ .$$

2. *For $r_0 \leq \Psi_1 \leq \Psi_0$, the constant crash hedging strategy is given by*

$$\hat{\varphi} = \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\,(\Psi_0 - \Psi_1)},$$

which is independent of the investment horizon $[t, T]$. The best constant worst case portfolio strategy is

$$\bar{\varphi}_t = \min\left(\left[\frac{1}{2}\left(\pi_0^* + \frac{1}{k^*}\right) - \sqrt{\frac{1}{4}\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{1}{\sigma_0^2\,(T-t)}}\right]^+ , \hat{\varphi}\right).$$

3. *For $\Psi_1 < r_0$ the constant crash hedging strategy $\hat{\varphi}_t$ is the solution of*

$$\ln\left(1 - \hat{\varphi}_t k_*\right) = \left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}\left(\hat{\varphi}_t - \pi_0^*\right)^2\right](T-t)\,.$$

This portfolio strategy is also the best constant worst case portfolio strategy, thus $\hat{\varphi}_t = \bar{\varphi}_t$.

Proof:

to 1: It remains to show that the best constant worst case portfolio strategy can not become negative. However, the expected worst case utility function of the investor has for $\varphi \leq 0$ the appearance

$$v_-(\varphi) := \ln(x) + \left(\Psi_0 - \frac{\sigma_0^2}{2}\left(\varphi - \pi_0^*\right)^2\right)(T-t)\,.$$

It is straightforward to verify, that $\pi_B \equiv 0$ maximizes this function for $\varphi \leq 0$, since $\pi_0^* \geq 0$. This concludes the first statement.

to 2: Follows analog to the above.

to 3: It remains to show that the constant crash hedging strategy is already the best constant worst case portfolio strategy if $\Psi_1 < r_0$. Since $\bar{\nu}_1\left(t, x\left(1 - \varphi k_*\right)\right)$ is decreasing in φ, it follows that $\bar{\varphi}_t \leq \hat{\varphi}_t$. However, any $\varphi < \hat{\varphi}_t$ gives a lower worst case bound than $\hat{\varphi}_t$, since $v_-(\varphi)$ is monotonously increasing in φ for $\varphi \leq 0$ (and $\pi_0^* \geq 0$). Thus, $\bar{\varphi}_t = \hat{\varphi}_t$.

This proves the proposition. □

For $\mu_0 < r_0$ one gets the following proposition.

Proposition 2.16
Let be $\mu_0 < r_0$. With the notation of Proposition 2.15, the following is valid.

1. *For $\Psi_1 > \Psi_0$ there exist no constant crash hedging strategy. However, the best constant worst case portfolio strategy is π_0^*.*
2. *For $r_0 \leq \Psi_1 \leq \Psi_0$ there exit a constant crash hedging strategy only if*
$$\ln\left(1 - \pi_0^* k_*\right) \leq \left(\Psi_0 - \Psi_1\right)\left(T - t\right).$$
In this case, the constant crash hedging strategy is given by $\hat{\varphi}_t = \varphi_t$, where φ_t is the solution of
$$\ln\left(1 - \varphi_t k_*\right) = \left(\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}\left(\varphi_t - \pi_0^*\right)^2\right)\left(T - t\right),$$
where the solution is chosen which is less or equal than π_0^*. The best constant worst case portfolio strategy is given by
$$\bar{\varphi}_t := \left\{ \begin{array}{ll} \hat{\varphi}_t & \textit{if } \ln\left(1 - \pi_0^* k_*\right) \leq \left(\Psi_0 - \Psi_1\right)\left(T - t\right) \\ \pi_0^* & \textit{if } \ln\left(1 - \pi_0^* k_*\right) > \left(\Psi_0 - \Psi_1\right)\left(T - t\right) \end{array} \right\}.$$
3. *For $\Psi_1 < r_0$ the constant crash hedging strategy is given implicitly by*
$$\ln\left(1 - \hat{\varphi}_t k_*\right) = \left(\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}\left(\hat{\varphi}_t - \pi_0^*\right)^2\right)\left(T - t\right).$$
The best constant worst case portfolio strategy is $\bar{\varphi}_t = \min\left(\hat{\varphi}_t, \pi_0^*\right)$.

Proof:

to 1: It is clear that there exists no crash hedging strategy in this case. The worst case is that no crash happens. Since
$$v_-(\varphi) := \ln(x) + \left(\Psi_0 - \frac{\sigma_0^2}{2}\left(\varphi - \pi_0^*\right)^2\right)\left(T - t\right),$$
it is straightforward that the best constant worst case portfolio strategy is $\hat{\varphi}_t = \pi_0^*$.

to 2: In this situtation, either no crash or a crash at the begining is the worst case. The defining equation for the crash hedging strategy is
$$v_-(\varphi) \quad = \quad \bar{\nu}_1(x, t)$$

$$\begin{aligned}
\iff \qquad & \ln(x) + \left[\Psi_0 - \frac{\sigma_0^2}{2}(\varphi - \pi_0^*)^2\right](T-t) \\
&= \ln(x) + \ln(1-\varphi k_*) + \Psi_1 (T-t) \\
\iff \qquad \ln(1-\varphi k_*) &= \left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}(\varphi - \pi_0^*)^2\right](T-t)\,.
\end{aligned}$$

The last equation has a solution if and only if

$$\ln(1 - \pi_0^* k_*) \le (\Psi_0 - \Psi_1)(T-t)\,.$$

In the case that equality holds there exists exactly one solution, namely $\hat{\varphi} = \pi_0^*$. If the inequality holds there exists two solutions with $\varphi_1 > \pi_0^* > \varphi_2$. Because of the monotonicity of $\bar{\nu}_1$ the rational choice is $\hat{\varphi} = \varphi_2$.

For $\ln(1-\pi_0^* k_*) > (\Psi_0 - \Psi_1)(T-t)$ there exist no constant crash hedging strategy, since

$$\ln(1-\varphi k_*) > \left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}(\varphi - \pi_0^*)^2\right](T-t) \quad \text{for all } \varphi \le 0.$$

In this situation the worst case is given by no crash. Hence, the best constant worst case portfolio strategy is $\bar{\varphi} = \pi_0^*$ as it has been shown above. Since $\nu_-(\varphi)$ is strictly increasing for $\varphi < \pi_0^*$, it is straightforward that $\bar{\varphi}_t = \hat{\varphi}_t$, if a constant crash hedging strategy exists.

to 3: For $\Psi_1 < r_0$, it is straightforward to verify that the equation

$$\ln(1-\varphi k_*) = \left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2}(\varphi - \pi_0^*)^2\right](T-t)$$

has a unique solution, which is equivalent to $\hat{\varphi}_t$. Moreover,

$$\bar{\varphi}_t = \left\{ \begin{array}{ll} \hat{\varphi}_t & \text{if } \hat{\varphi}_t < \pi_0^* \\ \pi_0^* & \text{else} \end{array} \right\}.$$

This is due to the fact that $\bar{\nu}_\varphi$ assumes it maximum in π_0^* and is strictly increasing for $\varphi < \pi_0^*$, and that $\bar{\nu}_1$ is decreasing in φ.

This concludes the proposition. $\square$

2.5 The Investor with Blurred Information

Assume now that the investor knows the market coefficient of the initial market 0 (e.g. by observation or by estimation). However, the inverstor does not know the market coefficients after a possible crash. Instead he does only know possible ranges for the market coefficients after a possible crash. More specific, let us suppose that the investor thinks that the market coefficients of market 1, that is r_1, μ_1, and σ_1 will be within the range of $[r_{1*}, r_1^*]$, $[\mu_{1*}, \mu_1^*]$, and $[\sigma_{1*}, \sigma_1^*]$ with $\sigma_{1*} > 0$, respectively. An investor with such information will be called **an investor with blurred information** and his crash hedging strategy will be named $\hat{\pi}_{bi}$ (where the bi in the subscript stands for blurred information). Hence, the **worst case scenario portfolio problem of the investor with blurred information** is

$$\inf_{\substack{r_1 \in [r_{1*}, r_1^*], \\ \mu_1 \in [\mu_{1*}, \mu_1^*], \sigma_1 \in [\sigma_{1*}, \sigma_1^*]}} \sup_{\pi(\cdot) \in A(x)} \inf_{\substack{0 \le \tau \le T, \\ k \in K}} \mathbb{E}\left[\ln\left(X^\pi(T)\right)\right] . \tag{31}$$

Proposition 2.17

1. *If $\Psi_{1_{\min}} \ge r_0$, then there exists a unique crash hedging strategy $\hat{\pi}_{bi}$, which is given by the solution of the differential equation*

$$\begin{aligned} \hat{\pi}'_{bi}(t) &= \left(\hat{\pi}_{bi}(t) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}_{bi}(t) - \pi_0^*\right)^2 + \Psi_{1_{\min}} - \Psi_0\right], && (32) \\ \textit{and} \quad \hat{\pi}_{bi}(T) &= 0. && (33) \end{aligned}$$

Moreover, this crash hedging strategy is bounded by $0 \le \hat{\pi}_{bi} \le \frac{1}{k^*}$, if $\Psi_{1_{\min}} > \Psi_0$. In the case of $\Psi_{1_{\min}} \le \Psi_0$, the crash hedging strategy is additionally bounded by

$$0 \le \hat{\pi}_{bi} \le \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_{1_{\min}}\right)}.$$

2. *If $\Psi_{1_{\min}} < r_0$, then there exists a unique crash hedging strategy $\hat{\pi}_{bi}$, which is given by the solution of the differential equation*

$$\begin{aligned} \hat{\pi}'_{bi}(t) &= \left(\hat{\pi}_{bi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}_{bi}(t) - \pi_0^*\right)^2 + \Psi_{1_{\min}} - \Psi_0\right], && (34) \\ \textit{and} \quad \hat{\pi}_{bi}(T) &= 0. && (35) \end{aligned}$$

Furthermore, this crash hedging strategy is bounded by

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_{1_{\min}}\right)} \le \hat{\pi}_{bi}(t) < 0 \qquad \textit{for } t \in [0, T).$$

3. *If $\Psi_{1_{\min}} < \Psi_0$ and $\pi_0^* < 0$, there exists a partial crash hedging strategy $\tilde{\pi}_{bi}$ (which is different from $\hat{\pi}_{bi}$), if*

$$S := T - \frac{\ln\left(1 - \pi_0^* k_*\right)}{\Psi_0 - \Psi_{1_{\min}}} > 0. \tag{36}$$

With this, $\tilde{\pi}_{bi}$ is on $[0,S]$ given by the unique solution of the differential equation

$$\begin{aligned} \tilde{\pi}_{bi}'(t) &= \left(\tilde{\pi}_{bi}(t) - \frac{1}{k_*}\right)\left[\frac{\sigma_0^2}{2}\left(\tilde{\pi}_{bi}(t) - \pi_0^*\right)^2 + \Psi_{1_{\min}} - \Psi_0\right], \quad (37) \\ \text{and} \quad \tilde{\pi}_{bi}(S) &= \pi_0^*. \quad (38) \end{aligned}$$

On $[S,T]$ set $\tilde{\pi}_{bi}(t) := \pi_0^*$. This partial crash hedging strategy is bounded by

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_{1_{\min}}\right)} \leq \tilde{\pi}_{bi} \leq \pi_0^* < 0.$$

The optimal portfolio strategy for an investor, who wants to maximize his worst case scenario portfolio problem, is given by

$$\bar{\pi}_{bi}(t) := \min\left\{\hat{\pi}_{bi}(t), \tilde{\pi}_{bi}(t), \pi_0^*\right\}, \quad (39)$$

where $\tilde{\pi}_{bi}(t)$ is only taken into account if it exists. $\bar{\pi}_{bi}$ will be denoted the **optimal crash hedging strategy** *of the investor with blurred information.*

Proof: Considering the market 1, which eventually reigns after a crash, the worst case for an investor is that Ψ_1 – the utility growth potential – will be minimal. Defining

$$\Psi_{1_{\min}} := \min\left\{\Psi_1 \;\middle|\; r_1 \in [r_{1*}, r_1^*], \mu_1 \in [\mu_{1*}, \mu_1^*], \sigma_1 \in [\sigma_{1*}, \sigma_1^*]\right\},$$

the proof follows now as the proof of Theorem 2.5. □

A special case is the **clueless** investor. The clueless investor has only a notion of what the interest rate might at least be. However, the clueless investor has no idea about neither the expected rate of return nor the volatility. Hence, in this situation the range of the market coefficients are $r_1 \in [r_{1*}, \infty)$, $\mu_1 \in \mathbb{R}$, and $\sigma_1 \in (0, \infty)$. It is straightforward to verify that the minimal utility growth potential in market 1 is given by $\Psi_{1_{\min}} = r_{1*}$. Thus the crash hedging strategy of the clueless investor which will be named $\hat{\pi}_{cl}$ (where the *cl* in the subscript stands for clueless) calculates as in Proposition 2.17, but with $\Psi_{1_{\min}} = r_{1*}$.

Remark 2.18
Note that r_{1*} can be either positive or negative. However, the cases $r_{1*} = 0$ and $r_{1*} = r_0$ are probably the most important ones.

1. Given that the clueless investor assumes that $r_{1*} = r_0$, which implies that $\hat{\pi}_{cl} \equiv 0$, this theory can explain why most people are not investing into the stock market. *No other portfolio theory can explain this fact.*

   However, if the clueless investor supposes that $r_{1*} = 0 < r_0$, which implies that $\hat{\pi}_{cl} < 0$, the clueless investor should go short in the stock market. This can not be observed in practise.

2. Also if the case $\hat{\pi}_{cl} > \pi_0^*$ is for $\pi_0^* > 0$ theoretical possible, it is practically irrelevant. Moreover, in general, it is valid that $\hat{\pi}_{cl} \ll \pi_0^*$.

2.6 Costs and Benefits of the Crash Hedging

This section discusses the costs and the potential benefits of the crash hedging strategy $\hat{\pi}$. It is important to keep in mind that all costs and benefits will be stated in terms of utility, *not* in terms of prices.

In the following, let us consider only the case $\Psi_1 \geq r_0$ and $\pi_0^* \geq 0$. However, the following applies basically also for the case $\Psi_1 < r_0$ and/or $\pi_0^* < 0$, if one exchanges k^* with k_* if necessary.

Definition 2.19

1. *The* **hedging cost $c(t)$** *against a crash are defined by*

$$c(t) := \nu_0\left(t, x\right) - \hat{\nu}\left(t, x\right),$$

where the costs are given by the loss in utility by using the crash indifferent strategy $\hat{\pi}$ instead of the classical strategy π_0^*.

2. *On the other hand, there are also benefits by taking the crash hedging strategy $\hat{\pi}$. These benefits are given by the potential loss in utility by a worst possible crash if the investor uses the classical optimal portfolio strategy π_0^* instead of the crash hedging strategy $\hat{\pi}$. Hence, the* **potential benefit $b(t)$** *is*

$$b(t) := \nu_0\left(t, x\right) - \nu_1\left(t, x\left(1 - \pi_0^* k^*\right)\right).$$

Remark 2.20

A rational investor who ignores the threat of a crash, uses the well–known classical optimal portfolio strategy π_0^*. The potential costs of ignoring a possible crash are given by $b(t)$, since this is the loss in utility given the portfolio strategy π_0^* and a crash of size k^* at time t. Defining

$$b(t;\pi) := \nu_\pi\left(t, x\right) - \nu_1\left(t, x\left(1 - \pi(t) k^*\right)\right),$$

this is the potential loss in utility, if the investor uses the nonnegative portfolio strategy π and a crash of size k^* happens at time t. Note that

$$b(t;\hat{\pi}) = \hat{\nu}\left(t, x\right) - \nu_1\left(t, x\left(1 - \hat{\pi}(t) k^*\right)\right) = 0.$$

Moreover, the potential benefit of using the crash hedging strategy $\hat{\pi}$ instead of the classical optimal portfolio strategy π_0^* is

$$b(t;\pi_0^*) - b(t;\hat{\pi}) = b(t) - 0 = b(t).$$

Hence, it is justified calling $b(t)$ the potential benefit of using $\hat{\pi}$ instead of π_0^*. Observe that the investor only gets the potential benefit, if a worst possible crash (namely of size k^*) happens. If only a crash of size k happens, the benefit is

$$b_k(t) := \nu_0(t,x) - \nu_1(t, x(1-\pi_0^* k)).$$

Obviously, if no crash happens, there is no benefit.

The hedging cost function $c(t)$ can be calculated to

$$\begin{aligned} c(t) &= \ln(x) + \Psi_0(T-t) - \ln(x) - \int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}(\hat{\pi}(s) - \pi_0^*)^2\right] ds \\ &= \frac{\sigma_0^2}{2}\int_t^T (\hat{\pi}(s) - \pi_0^*)^2\, ds, \end{aligned}$$

which shows that the costs of the crash hedging are independent of the wealth of the investor. Hence, the hedging cost function $c(t)$ is well defined. Notice that equation (12) implies

$$c(t) = (\Psi_0 - \Psi_1)(T-t) - \ln(1 - \hat{\pi}(t)k^*). \tag{40}$$

Taking the derivative of c with respect to t gives the **marginal hedging costs**

$$c'(t) = -\frac{\sigma_0^2}{2}(\hat{\pi}(t) - \pi_0^*)^2,$$

which are always decreasing in t.

Moreover, the potential benefit function $b(t)$ can be calculated explicitly to

$$\begin{aligned} b(t) &= \ln(x) + \Psi_0(T-t) - [\ln(x) + \ln(1-\pi_0^* k^*) + \Psi_1(T-t)] \\ &= (\Psi_0 - \Psi_1)(T-t) - \ln(1 - \pi_0^* k^*), \end{aligned}$$

which also shows that b is well defined, since the benefit is independent of the wealth of the investor. The **marginal potential benefits** are given by

$$b'(t) = -(\Psi_0 - \Psi_1),$$

which can be zero (case i)), negative (case ii)), or positive (case iii)). The following lemma can be formulated now.

Lemma 2.21
The **potential net benefit $n(t)$** *is given by*

$$n(t) = \ln\left(\frac{1 - \hat{\pi}(t)k^*}{1 - \pi_0^* k^*}\right).$$

Every crash hedging strategy $\hat{\pi}$ will be pursuit by the rational investor only if $\hat{\pi}(t) \leq \pi_0^*$.

Proof: The potential net benefit calculates to

$$\begin{aligned} n(t) &= b(t) - c(t) \\ &= \ln\left(1 - \hat{\pi}(t)k^*\right) - \ln\left(1 - \pi_0^* k^*\right). \end{aligned}$$

Since

$$n(t) \begin{cases} > 0 & if \quad \hat{\pi} < \pi_0^* \\ < 0 & if \quad \hat{\pi} > \pi_0^*, \end{cases}$$

any crash hedging strategy $\hat{\pi}$ which is larger than π_0^* is disadvantageous for the investor. The crash hedging strategy $\hat{\pi}$ is advantageous to an investor, who wants to maximize her worst case utility, if $\hat{\pi} < \pi_0^*$. Hence, any rational investor will pursuit the crash hedging strategy $\hat{\pi}$ only if it is less or equal than the classical optimal portfolio strategy π_0^*. □

Note that the net benefit only depends on the initial market conditions (via π_0^*) and is independent of the market conditions after a possible crash.

Remark 2.22
Observe that equation (40) can also be derived as follows

$$\begin{aligned} c(t) &= \nu_0\left(t, x\right) - \hat{\nu}\left(t, x\right) \\ &= \nu_0\left(t, x\right) - \nu_1\left(t, x\left(1 - \hat{\pi}(t)k^*\right)\right) \\ &= \left(\Psi_0 - \Psi_1\right)\left(T - t\right) - \ln\left(1 - \hat{\pi}(t)k^*\right). \end{aligned}$$

With this, the marginal hedging costs are

$$c'(t) = -\left(\Psi_0 - \Psi_1\right) + \frac{\hat{\pi}'(t)k^*}{1 - \hat{\pi}(t)k^*},$$

where the equivalence of these marginal costs with the marginal costs calculated above are also guaranteed by equation (13).

Finally, let us calculate the costs of having only blurred information. Therefore, consider the costs of using the crash hedging strategy $\hat{\pi}_{bi}$

$$c_{bi}(t) := \left(\Psi_0 - \Psi_{1_{\min}}\right)\left(T - t\right) - \ln\left(1 - \hat{\pi}_{bi}(t)k^*\right).$$

Hence, the costs of having only blurred information are

$$c_{bi}(t) - c(t) = \ln\left(\frac{1 - \hat{\pi}(t)k^*}{1 - \hat{\pi}_{bi}(t)k^*}\right) + \left(\Psi_1 - \Psi_{1_{\min}}\right)\left(T - t\right).$$

In general, these costs are positive. More specific, these costs are positive, if $\hat{\pi}_{bi} \leq \hat{\pi}$, or if

$$\left(\Psi_1 - \Psi_{1_{\min}}\right)\left(T - t\right) > -\ln\left(\frac{1 - \hat{\pi}(t)k^*}{1 - \hat{\pi}_{bi}(t)k^*}\right).$$

2.7 Arbitrary Number of Crashes

Let us assume there are n crashes expected within the time horizon T. The market coefficient after the j–th crash will be denoted by r_j, μ_j, and σ_j. Again, for simplicity, this will be called the market j. In accordance with Definition 2.3 π_j^* and Ψ_j are defined likewise. Moreover, let us suppose that the j–th crash ranges in size between k_{*j} and k_j^*, with $0 < k_{*j} < k_j^* < 1$. The crash hedging strategy for market j will be called $\hat{\pi}_j$. Furthermore, setting $\hat{\pi}_n := \pi_n^*$ (since no more crash is expected, this is justified), one has the following theorem.

Theorem 2.23
For $j = 0, \ldots, n-1$, there exists unique crash hedging strategies $\hat{\pi}_j$.

1. *If $\Psi_{j+1} \geq r_j$, then $\hat{\pi}_j$ is given by the solution of the differential equation*

$$\begin{aligned} \hat{\pi}_j'(t) &= \left(\hat{\pi}_j(t) - \frac{1}{k_j^*}\right)\left[\frac{\sigma_j^2}{2}\left(\hat{\pi}_j(t) - \pi_j^*\right)^2\right. && (41) \\ &\qquad \left. - \frac{\sigma_{j+1}^2}{2}\left(\hat{\pi}_{j+1}(t) - \pi_{j+1}^*\right)^2 + \Psi_{j+1} - \Psi_j\right], \\ \textit{and} \quad \hat{\pi}_j(T) &= 0. && (42) \end{aligned}$$

   Moreover, these crash hedging strategies are bounded by $0 \leq \hat{\pi}_j < \frac{1}{k_j^*}$, if $\Psi_{j+1} > \Psi_j$. In the case of $\Psi_{j+1} \leq \Psi_j$, the crash hedging strategy is additionally bounded by

$$0 \leq \hat{\pi}_j < \pi_j^* - \sqrt{\frac{2}{\sigma_j^2}\left(\Psi_j - \Psi_{j+1}\right)}.$$

2. *If $\Psi_{j+1} < r_j$, then $\hat{\pi}_j$ is given by the solution of the differential equation*

$$\begin{aligned} \hat{\pi}_j'(t) &= \left(\hat{\pi}_j(t) - \frac{1}{k_{*j}}\right)\left[\frac{\sigma_j^2}{2}\left(\hat{\pi}_j(t) - \pi_j^*\right)^2\right. && (43) \\ &\qquad \left. - \frac{\sigma_{j+1}^2}{2}\left(\hat{\pi}_{j+1}(t) - \pi_{j+1}^*\right)^2 + \Psi_{j+1} - \Psi_j\right], \\ \textit{and} \quad \hat{\pi}_j(T) &= 0. && (44) \end{aligned}$$

   Furthermore, this crash hedging strategy is bounded by

$$\pi_j^* - \sqrt{\frac{2}{\sigma_j^2}\left(\Psi_j - \Psi_{j+1}\right)} < \hat{\pi}_j(t) < 0 \quad \textit{for } t \in [0, T).$$

3. *If $\Psi_{j+1} < \Psi_j$ and $\pi_j^* < 0$, there exists a partial crash hedging strategy $\tilde{\pi}_j$ (which is different from $\hat{\pi}_j$), if*

$$S_j := T - \frac{\ln\left(1 - \pi_j^* k_{*j}\right)}{\Psi_j - \Psi_{j+1}} > 0. \tag{45}$$

With this, $\tilde{\pi}_j$ is on $[0, S_j]$ given by the unique solution of the differential equation

$$\begin{aligned} \tilde{\pi}_j'(t) &= \left(\tilde{\pi}_j(t) - \frac{1}{k_{*j}}\right)\left[\frac{\sigma_j^2}{2}\left(\hat{\pi}_j(t) - \pi_j^*\right)^2\right. &(46)\\ &\qquad \left. - \frac{\sigma_{j+1}^2}{2}\left(\bar{\pi}_{j+1}(t) - \pi_{j+1}^*\right)^2 + \Psi_{j+1} - \Psi_j\right], \\ \text{and} \quad \tilde{\pi}_j(S_j) &= \pi_j^*. &(47) \end{aligned}$$

On $[S_j, T]$ set $\tilde{\pi}_j(t) := \pi_j^*$. This partial crash hedging strategy is bounded by

$$\pi_j^* - \sqrt{\frac{2}{\sigma_j^2}\left(\Psi_j - \Psi_{j+1}\right)} < \tilde{\pi}_j \le \pi_j^* < 0.$$

The optimal portfolio strategy for an investor, who wants to maximize her worst case scenario portfolio problem, is given by

$$\bar{\pi}_j(t) := \min\left\{\hat{\pi}_j(t), \tilde{\pi}_j(t), \pi_j^*\right\}, \tag{48}$$

where $\tilde{\pi}_j(t)$ is only taken into account if it exists. $\bar{\pi}_j$ is denoted the **optimal crash hedging strategy in market *j***.

Proof: Obviously, for $j = n-1$ the above statement corresponds to Theorem 2.5. Hence, assume that $j < n-1$. Let be $\Psi_{j+1} \ge r_j$ and let $\hat{\nu}_j$ be the corresponding value function of the portfolio strategy $\hat{\pi}_j$. The necessary condition for a crash hedging strategy is then

$$\hat{\nu}_j\left(t, x\right) = \hat{\nu}_{j+1}\left(t, x\left(1 - \hat{\pi}_j(t) k_j^*\right)\right).$$

This equation states that the expected utility should be equal, no matter whether a crash of the worst possible size happens or not. Using the definition of $\hat{\nu}$ and further transforming gives

$$\begin{aligned} \ln(x) + \int_t^T \left[\Psi_j - \frac{\sigma_j^2}{2}\left(\hat{\pi}_j(s) - \pi_j^*\right)^2\right] ds & \\ = \ln(x) + \ln\left(1 - \hat{\pi}_j(t) k_j^*\right) & \end{aligned}$$

$$+\int_t^T \left[\Psi_{j+1} - \frac{\sigma_{j+1}^2}{2}\left(\hat{\pi}_{j+1}(s) - \pi_{j+1}^*\right)^2\right] ds$$

$$\iff \quad \ln\left(1 - \hat{\pi}_j(t)k_j^*\right) = \int_t^T \left[\Psi_j - \Psi_{j+1} - \frac{\sigma_j^2}{2}\left(\hat{\pi}_j(s) - \pi_j^*\right)^2 + \frac{\sigma_{j+1}^2}{2}\left(\hat{\pi}_{j+1}(s) - \pi_{j+1}^*\right)^2\right] ds.$$

Differentiation yields now

$$\frac{-\hat{\pi}_j'(t)k_j^*}{1 - \hat{\pi}_j(t)k_j^*} = \Psi_{j+1} - \Psi_j + \frac{\sigma_j^2}{2}\left(\hat{\pi}_j(t) - \pi_j^*\right)^2 - \frac{\sigma_{j+1}^2}{2}\left(\hat{\pi}_{j+1}(t) - \pi_{j+1}^*\right)^2,$$

which gives the asserted differential equation. The differential equation for the case $\Psi_{j+1} < r_j$ can be derived analog. The remainder follows as in the proof of Theorem 2.5. □

2.8 Deterministic Portfolio Strategies

Definition 2.24
Let π be an admissible portfolio strategy.

$$\pi_d(t) := \mathbb{E}\left[\pi(t)\right] \quad \textit{for all } t \in [0, T]$$

will be called the **corresponding deterministic portfolio strategy to $\boldsymbol{\pi}$.**

Because of the construction of π_d there exist Ω_1 and Ω_2, which are subsets of Ω, such that

$$P(\Omega_1) > 0 \quad \text{and} \quad P(\Omega_2) > 0 \quad \text{and}$$
$$\pi(t, \omega_1) \le \pi_d(t) \le \pi(t, \omega_2) \quad \text{for all } \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2,\ \text{and } t \in [0, T].$$

This implies that π_d is admissible.

Definition 2.25
Let us define

$$k_\pi(t) := k^* \cdot \mathbb{1}_{\{\pi(t) \ge 0\}} + k_* \cdot \mathbb{1}_{\{\pi(t) < 0\}}.$$

Lemma 2.26
Let π be an admissible portfolio strategy. Then the corresponding deterministic portfolio strategy to π yields in the initial crash–free market at least the same expected final utility as π. If, additionally

$$A_\pi(t) := \ln\left(1 - \mathbb{E}\left[\pi(t)\right] k_{\pi_d}(t)\right) - \mathbb{E}\left[\ln\left(1 - \pi(t)k_\pi(t)\right)\right]$$

$$\geq \quad 0$$

holds, then π_d yields in the initial market with a possible crash at least the same worst case expected final utility as π.

Proof: Using the Theorem of Fubini, one has for any admissible portfolio strategy π

$$\begin{aligned}
\nu_\pi(t,x) &= \ln(x) + \mathbb{E}\left[\int_t^T \Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2 \, ds\right] \\
&= \ln(x) + \int_t^T \Psi_0 - \frac{\sigma_0^2}{2}\mathbb{E}\left[\left(\pi(s) - \pi_0^*\right)^2\right] \, ds \\
&= \ln(x) + \int_t^T \Psi_0 - \frac{\sigma_0^2}{2}\left(\mathbb{E}\left[\pi(s)\right] - \pi_0^*\right)^2 - \frac{\sigma_0^2}{2}\mathrm{Var}\left(\pi(s)\right) \, ds \\
&= \ln(x) + \int_t^T \Psi_0 - \frac{\sigma_0^2}{2}\left(\pi_d(s) - \pi_0^*\right)^2 - \frac{\sigma_0^2}{2}\mathrm{Var}\left(\pi(s)\right) \, ds \\
&= \nu_{\pi_d}(t,x) - \frac{\sigma_0^2}{2}\mathrm{Var}\left(\pi(s)\right) \, ds \\
&\leq \nu_{\pi_d}(t,x).
\end{aligned}$$

This is the case if no crash happens. In the case that a crash has happened, one gets with

$$A_\pi(t) = \ln\left(1 - \mathbb{E}\left[\pi(t)\right] k_{\pi_d}(t)\right) - \mathbb{E}\left[\ln\left(1 - \pi(t)k_\pi(t)\right)\right] \geq 0$$

the following

$$\begin{aligned}
\nu_1\left(t, x\left(1 - \pi(t)k_\pi(t)\right)\right) &= \ln(x) + \mathbb{E}\left[\ln\left(1 - \pi(t)k_\pi(t)\right)\right] + \Psi_1\left(T - t\right) \\
&= \ln(x) + \ln\left(1 - \pi_d(t)k_{\pi_d}(t)\right) + \Psi_1\left(T - t\right) - A_\pi(t) \\
&= \nu_1\left(t, x\left(1 - \pi_d(t)k_{\pi_d}(t)\right)\right) - A_\pi(t) \\
&\leq \nu_1\left(t, x\left(1 - \pi_d(t)k_{\pi_d}(t)\right)\right).
\end{aligned}$$

This proves the assertion. □

Remark 2.27
The condition $A_\pi(t) \geq 0$ holds for example if $\pi(t) \geq 0$ P-a.s. or if $\pi_d(t) < 0$.

2.9 Geometric Interpretation of the Crash Hedging Strategy

Let us consider only the case $\pi_0^* \geq 0$. Setting

$$B_\pi(t) := \frac{\sigma_0^2}{2} \int_t^T \operatorname{Var}(\pi(s))\, ds \geq 0$$

and using the Theorem of Fubini, ν_π can be rewritten as

$$\nu_\pi(t,x) = \ln(x) + \int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\mathbb{E}[\pi] - \pi_0^*\right)^2\right] ds - B_\pi(t),$$

as it has been seen in the proof of Lemma 2.26. There are two different possibilities to take into consideration.

1. $\left(\mathbb{E}[\pi(s)] - \pi_0^*\right)^2 < \left(\hat{\pi}(s) - \pi_0^*\right)^2$. This inequality is valid, if either

$$\begin{array}{lll} \text{i)}\ 0 \leq \mathbb{E}[\pi(s)] - \pi_0^* < \hat{\pi}(s) - \pi_0^* & \Longleftrightarrow & \pi_0^* \leq \mathbb{E}[\pi(s)] < \hat{\pi}(s) \ \text{or} \\ \text{ii)}\ 0 \geq \mathbb{E}[\pi(s)] - \pi_0^* > \hat{\pi}(s) - \pi_0^* & \Longleftrightarrow & \hat{\pi}(s) < \mathbb{E}[\pi(s)] \leq \pi_0^*. \end{array}$$

   Applying the above inequality yields

$$\begin{array}{rrcl} & \nu_\pi(t,x) & > & \hat{\nu}(t,x) - B_\pi(t) \\ \Longrightarrow & \nu_\pi(t,x) & \geq & \hat{\nu}(t,x), \end{array}$$

   where the last inequality holds only if the variance of π is small enough. Thus, this inequality holds e.g. for all deterministic portfolio strategies, which obviously satisfy $B_\pi \equiv 0$.

2. $\left(\mathbb{E}[\pi(s)] - \pi_0^*\right)^2 > \left(\hat{\pi}(s) - \pi_0^*\right)^2$. This inequality is valid, if either

$$\begin{array}{lll} \text{iii)}\ \mathbb{E}[\pi(s)] - \pi_0^* > \hat{\pi}(s) - \pi_0^* \geq 0 & \Longleftrightarrow & \mathbb{E}[\pi(s)] > \hat{\pi}(s) \geq \pi_0^* \ \text{or} \\ \text{iv)}\ \mathbb{E}[\pi(s)] - \pi_0^* < \hat{\pi}(s) - \pi_0^* \leq 0 & \Longleftrightarrow & \mathbb{E}[\pi(s)] < \hat{\pi}(s) \leq \pi_0^*. \end{array}$$

   Using the above inequality gives

$$\begin{array}{rcl} \nu_\pi(t,x) & < & \hat{\nu}(t,x) - B_\pi(t) \\ & \leq & \hat{\nu}(t,x). \end{array}$$

In order to analyse the situation after a possible crash, let us use

$$\nu_1\left(t, x\left(1 - \pi(t)k_\pi(t)\right)\right) = \ln(x) + \ln\left(1 - \mathbb{E}[\pi(t)]\, k_{\pi_d}(t)\right) + \Psi_1(T-t) - A_\pi(t),$$

which has been derived in the proof of Lemma 2.26. There are two different cases to be regarded.

I) $\mathbb{E}[\pi(t)] > \hat{\pi}(t) \iff \ln(1-\mathbb{E}[\pi(t)]\,k_{\pi_d}(t)) < \ln(1-\hat{\pi}(t)k_{\hat{\pi}}(t))$, since this also implies that $k_{\pi_d}(t) \geq k_{\hat{\pi}}(t)$. Thus this case yields

$$\begin{aligned}\nu_1\left(t, x\left(1-\pi(t)k_\pi(t)\right)\right) &< \nu_1\left(t, x\left(1-\hat{\pi}(t)k_{\hat{\pi}}(t)\right)\right) - A_\pi(t)\\ &\leq \nu_1\left(t, x\left(1-\hat{\pi}(t)k_{\hat{\pi}}(t)\right)\right),\end{aligned}$$

given that $A_\pi(t) \geq 0$.

II) $\mathbb{E}[\pi(t)] < \hat{\pi}(t) \iff \ln(1-\mathbb{E}[\pi(t)]\,k_{\pi_d}(t)) > \ln(1-\hat{\pi}(t)k_{\hat{\pi}}(t))$, since this also implies that $k_{\pi_d}(t) \leq k_{\hat{\pi}}(t)$. Hence, this case gives

$$\begin{aligned}\nu_1\left(t, x\left(1-\pi(t)k_\pi(t)\right)\right) &> \nu_1\left(t, x\left(1-\hat{\pi}(t)k_{\hat{\pi}}(t)\right)\right) - A_\pi(t)\\ &\geq \nu_1\left(t, x\left(1-\hat{\pi}(t)k_{\hat{\pi}}(t)\right)\right),\end{aligned}$$

if either $A_\pi(t) \geq 0$ is small enough or $A_\pi(t) \leq 0$.

Let us dicuss Figure 1 by considering only deterministic portfolio strategies which is justified by Lemma 2.26. For deterministic portfolio strategies, B_π and A_π are equal to zero. Let us denote

$$\nu_{1,\pi}(t,x) := \nu_1\left(t, x\left(1-\pi(t)k_\pi(t)\right)\right).$$

An investor who pursues a portfolio strategy which lies in the area iv) (and thus area II)) is better of after an immediate crash compared to if no crash happens. To see this, case iv) gives $\nu_\pi < \hat{\nu}$ and case II) gives $\nu_{1,\pi} > \nu_{1,\hat{\pi}}$. Hence

$$\nu_\pi < \hat{\nu} = \nu_{1,\hat{\pi}} < \nu_{1,\pi},$$

which shows that this investor favors a crash.

On the contrary in the situation of ii) the investor is better of with no crash happening, since

$$\nu_\pi > \hat{\nu} = \nu_{1,\hat{\pi}} > \nu_{1,\pi}.$$

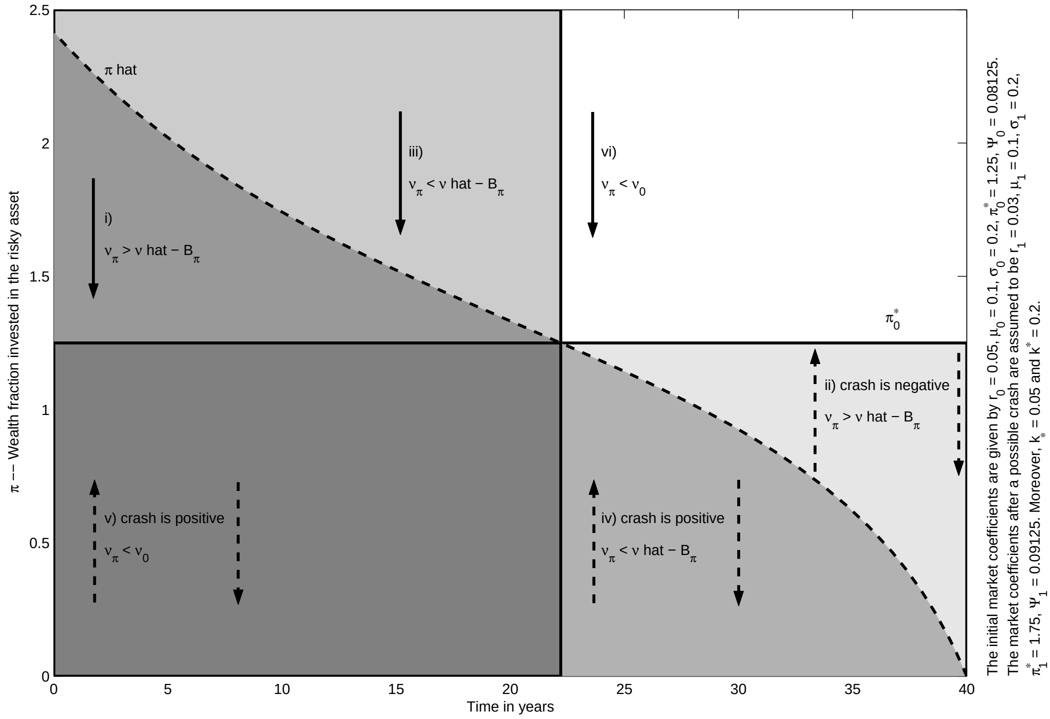
Being in the situation of iii) an investor will always have a higher utility by using the crash hedging strategy $\hat{\pi}$. This is, because case iii) delivers $\nu_\pi < \hat{\nu}$ and case I) yields $\nu_{1,\pi} < \nu_{1,\pi_0^*}$, where this last inequality is due to the monotonicity of the logarithm and thus of $\nu_{1,\pi}$ in π.

The remaining possibilities can only happen if $\Psi_1 > \Psi_0$. Using a strategy which lies in area iii) yields always a lower utility than using the crash hedging strategy. Being in area i), it is always better to use the classical crash–free optimal portfolio strategy π_0^*. This is due to the fact that $\nu_{1,\pi_0^*} > \nu_{1,\pi}$ and $\nu_0 > \nu_\pi$ for all π which lie in area i).

In order to analyse the situation v), it is important to realize that $\nu_0 \leq \nu_{1,\pi_0^*}$. Equality clearly holds in t_0, which is the upper right corner of area v) in Figure 1. Moreover, the inequality holds, since $\Psi_1 > \Psi_0$ in this situation. Hence,

$$\nu_\pi < \nu_0 = \nu_{1,\pi_0^*} < \nu_{1,\pi}.$$

Figure 1: Geometric Interpretation of the Crash Hedging Strategy



This graphic shows how ν_π and $\hat{\nu}$ relate to each other. Note that the areas i), iii), and v) can only occur if $\Psi_1 > \Psi_0$ and the time horizon is sufficient large. Moreover notice that the area I) is the union of the areas ii), iii) and vi) while the area II) is the union of the areas i), iv), and v). The solid arrows indicate the possibility to increase the expected utility in market 0 as well as in market 1 in the given area by simply reducing π. The first dashed arrow in an area indicates the possibility to increase the expected utility in market 0 by simply increasing π. The second dashed arrow in an area indicates the possibility to increase the expected utility in market 1 by simply reducing π.

No crash occurring will give the lowest expected utility for an investor being with his portfolio strategy in area v). This investor will favor a crash.

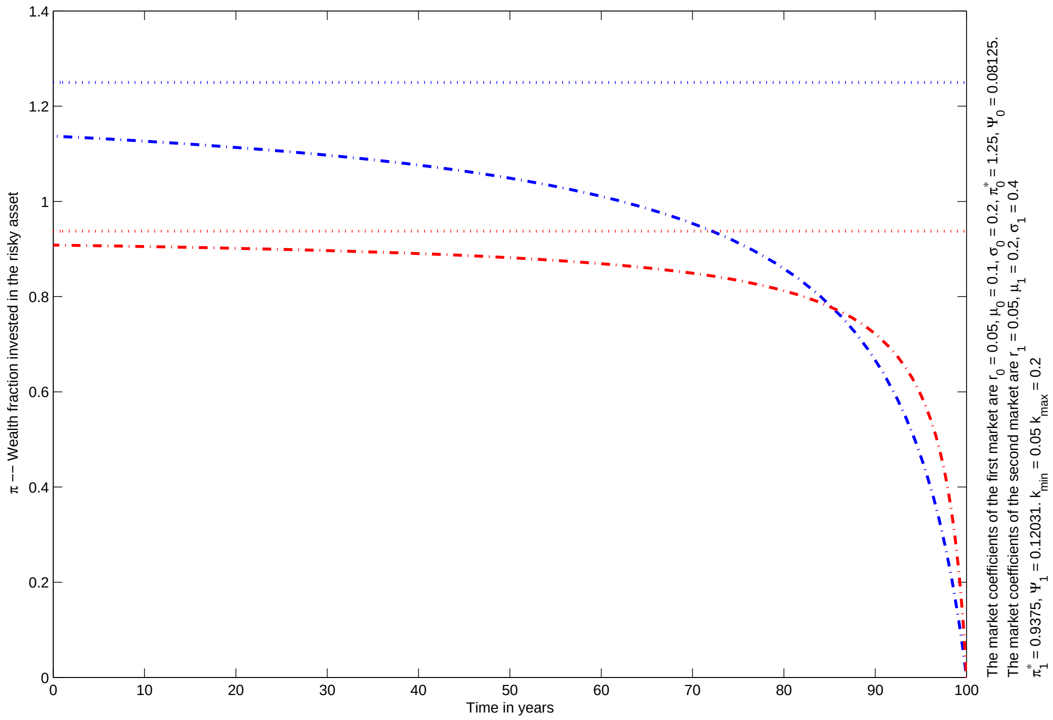
All in all , it has been shown that the investor who pursues a portfolio strategy π with $\pi < \min(\hat{\pi}, \pi_0^*)$ favors a crash, since deviate from $\min(\hat{\pi}, \pi_0^*)$ by reducing π increases the expected utility after a possible crash, but reduces the expected utility in the initial market. Using a strategy π with $\hat{\pi} < \pi < \pi_0^*$ will increase the expected utility in the initial market and reduce the expected utility after a possible crash (compared to the strategy $\min(\hat{\pi}, \pi_0^*)$). An investor using such a strategy favors that no crash happens and fears a possible crash. Using a strategy $\pi > \pi_0^*$ is never optimal.

For $\pi_0^* < 0$ the above geometric interpretation does not hold for $\hat{\pi}$, since the Bellman–principle is violated – $\hat{\pi}$ is no longer optimal on $[S, T]$ with S given by (17). However, similar results can be obtained for $\tilde{\pi}$ – the partial crash hedging strategy.

2.10 Examples and Further Remarks

Observe that it is possible that $\pi_0^* > \pi_1^*$, but $\hat{\phi}_0'(T) < \hat{\phi}_1'(T)$ and thus $\hat{\phi}_0(t) < \hat{\phi}_1(t)$ for $t \in [T-\epsilon, T]$ and for a suitable $\epsilon > 0$. However, if the time horizon T is long enough, it is valid that $\hat{\phi}_0(t) > \hat{\phi}_1(t)$ for some $t \in [0, \delta]$ with $\delta > 0$ being chosen suitable (see Figure 2 and Figure 8).

Figure 2: Example $\pi_0^* > \pi_1^*$, but $\hat{\phi}_0'(T) < \hat{\phi}_1'(T)$



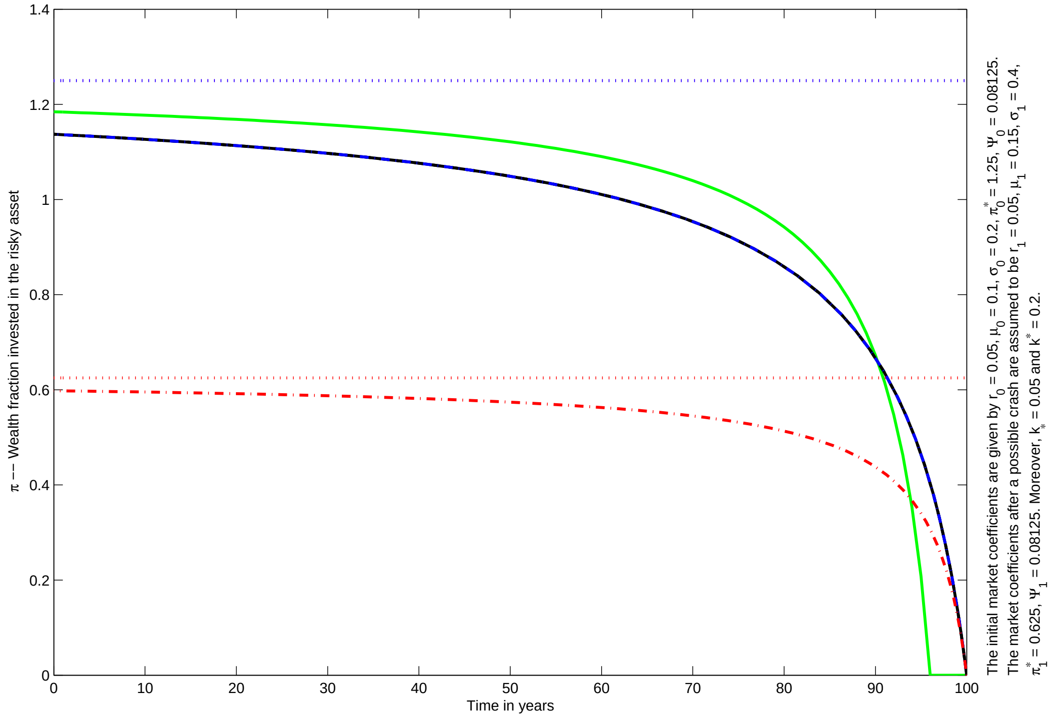
This graphic shows $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

1. $\boldsymbol{\Psi_1 = \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* \geq 0}$
   Be aware that this case includes the case of non–changing market coefficient (and it is not only this case). Moreover, this case is valid if the market conditions change in such a way that the utility growth potential does not change (see Figure 3). Note that in this case $\hat{\pi} = \hat{\phi}_0 \neq \hat{\phi}_1$. The last inequality is due to the fact that in general $\pi_0^* \neq \pi_1^*$.

   The constant crash hedging strategy is $\hat{\varphi} = \pi_0^*$. Notice that it is possible that the best constant worst case portfolio strategy $\bar{\varphi}_t$ is greater than $\bar{\pi}(t)$

if t is sufficiently small and T sufficient large (see Figure 3). However, close to the terminal investment period T it will always be less or equal than the continuous optimal crash hedging strategy.

Figure 3: Example $\Psi_1 = \Psi_0$ and $\pi_0^* \geq 0$



This graphic shows $\hat{\pi} = \bar{\pi} = \hat{\phi}_0$ (blue dash–dotted line with black background), $\hat{\varphi} = \pi_0^*$ (blue dotted line), $\bar{\varphi}$ (green line), $\hat{\phi}_1$ (red dash–dotted line), and π_1^* (red dotted line).

2. $\boldsymbol{\Psi_1 > \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* \geq 0}$

   There are several observations to make. First, note that the $\hat{\pi}$ in this case descents faster than $\hat{\phi}_0$. Thus, $\hat{\pi}(t) \geq \hat{\phi}_0(t)$ for all $t \in [0, T]$. This can also be verified in Figure 4. However, nothing comparable can be said about $\hat{\pi}$ and $\hat{\phi}_1$.

   According to Lemma 2.21 the *optimal crash hedging strategy* has to satisfy $\hat{\pi} \leq \pi_0^*$. However, in this case it is possible that the crash hedging strategy will become greater than π_0^* given that the time horizon is large enough and $\pi_0^* < \frac{1}{k^*}$. To analyze this, define

$$t_0 := T + \frac{\ln\left(1 - \pi_0^* k^*\right)}{\Theta_2} + \frac{\pi_0^* - \frac{1}{k^*}}{\Delta_1 \cdot C} \arctan\left(\frac{\pi_0^*}{\Delta_1}\right) - \frac{1}{2\Theta_2} \ln\left(\frac{\Delta_1^2}{(\pi_0^*)^2 + \Delta_1^2}\right)$$

with

$$\Delta_1 := \sqrt{\frac{2}{\sigma_0^2}(\Psi_1 - \Psi_0)} \qquad \text{and}$$

$$\Theta_2 := \frac{\sigma_0^2}{2}\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \Psi_1 - \Psi_0.$$

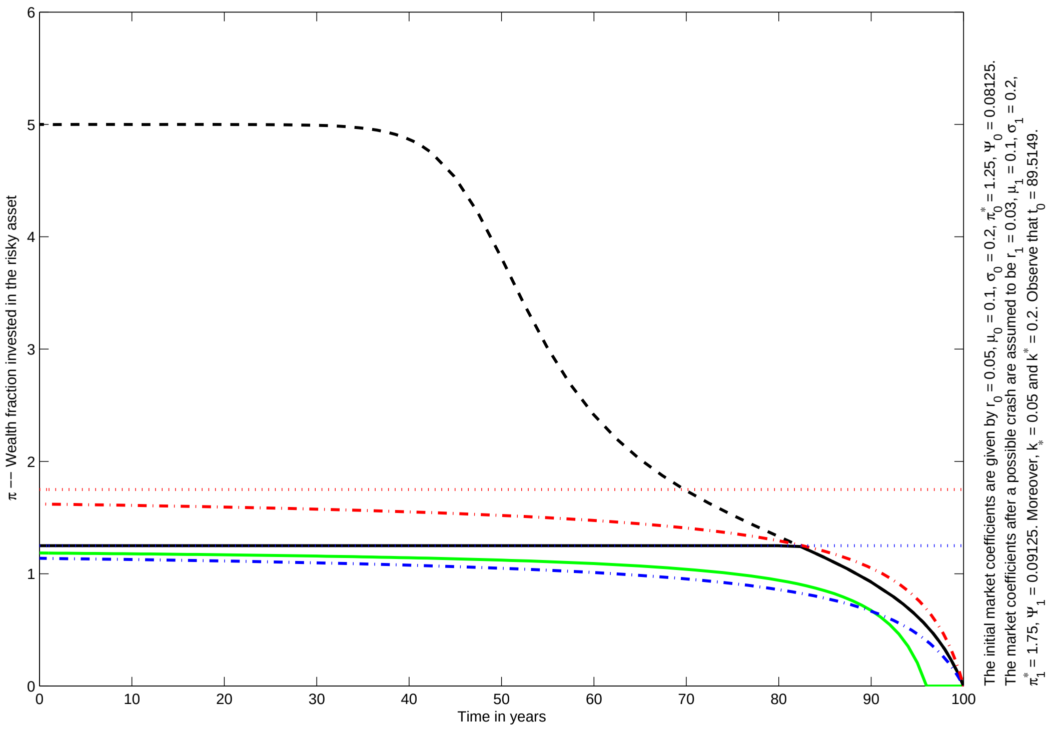
This point has been obtained by setting $\hat{\pi}(t_0) = \pi_0^*$ in equation (29) in Proposition 2.11. Hence, if $t_0 \in (0, T]$, then the optimal crash hedging strategy is

$$\bar{\pi}(t) := \begin{cases} \pi_0^*, & \text{for } t \leq t_0 \\ \hat{\pi}(t), & \text{for } t > t_0 \end{cases},$$

as it can be verified in Figure 4.

Here, no constant crash hedging strategy exists. However, the best constant worst case portfolio strategy is in the example always less than the continuous optimal crash hedging strategy. Though this is not generally true for this case.

Figure 4: Example $\Psi_1 > \Psi_0$ and $\pi_0^* \geq 0$



This graphic shows $\hat{\pi}$ (black dashed line), $\bar{\pi}$ (black line), $\bar{\varphi}$ (green line), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

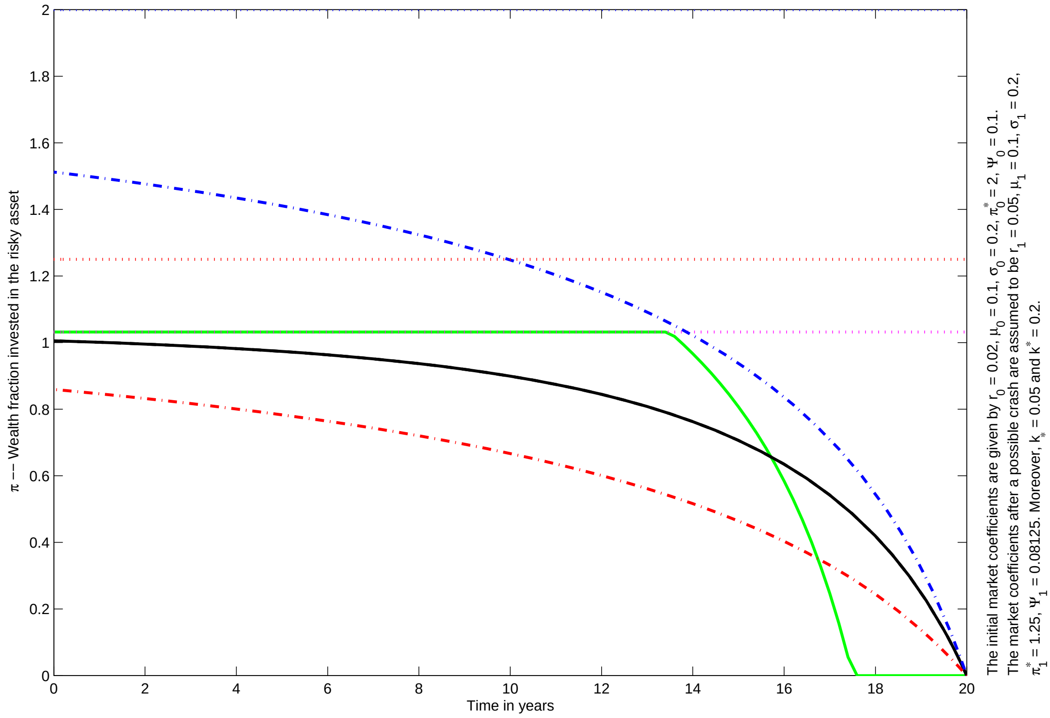
3. ***$r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$***

   Note that the $\hat{\pi}$ in this case descents slower than $\hat{\phi}_0$. This is, because the correction term $\Psi_1 - \Psi_0$ is negative. Thus, $\hat{\pi}(t) \leq \hat{\phi}_0(t)$ for all $t \in [0, T]$. This can also be verified in Figure 5 and Figure 6. However, nothing comparable can be said about $\hat{\pi}$ and $\hat{\phi}_1$.

   Moreover, observe that $0 \leq \hat{\pi}(t) < \min\left(\hat{\varphi}, \frac{1}{k^*}\right)$ for all $t \in [0, T]$, as it has been stated in Theorem 2.5. In particular, $\hat{\pi}(0) \longrightarrow \min\left(\hat{\varphi}, \frac{1}{k^*}\right)$ for $T \longrightarrow \infty$ (see Figure 6).

   The constant crash hedging strategy is $\hat{\varphi} = \pi_0^*$. Note that it is possible that the best constant worst case portfolio strategy $\bar{\varphi}_t$ is greater than $\bar{\pi}(t)$ if t is sufficiently small and the investment horizon T is sufficiently large (see Figure 5). Furthermore, $0 \leq \bar{\varphi} \leq \hat{\varphi}$ as it has been stated in Proposition 2.15.

Figure 5: Example $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$

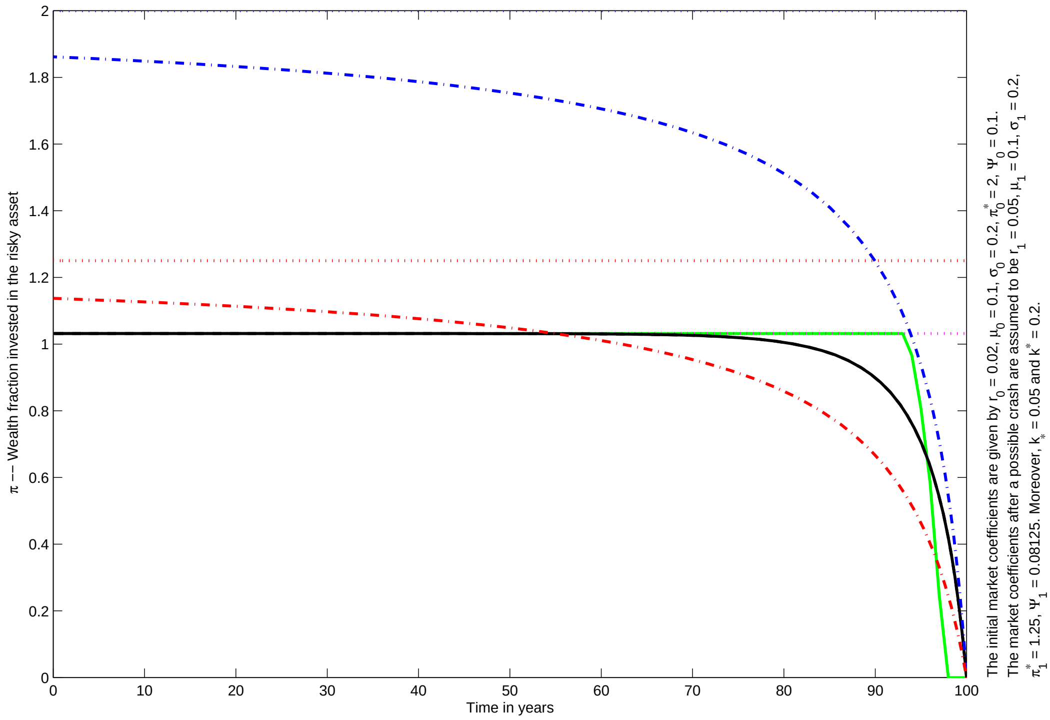


This graphic shows $\hat{\pi} = \bar{\pi}$ (black line), $\hat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), $\pi_0^* = 2$ (blue dotted line), and π_1^* (red dotted line).

4. ***$\Psi_1 < r_0$ and $\pi_0^* \geq 0$***

   The crash hedging strategy which is also an optimal crash hedging strategy

Figure 6: Example $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* \geq 0$, the long term behaviour



This graphic shows the long term behaviour of $\hat{\pi} = \bar{\pi}$ (black line), $\hat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), $\pi_0^* = 2$ (blue dotted line), and π_1^* (red dotted line).

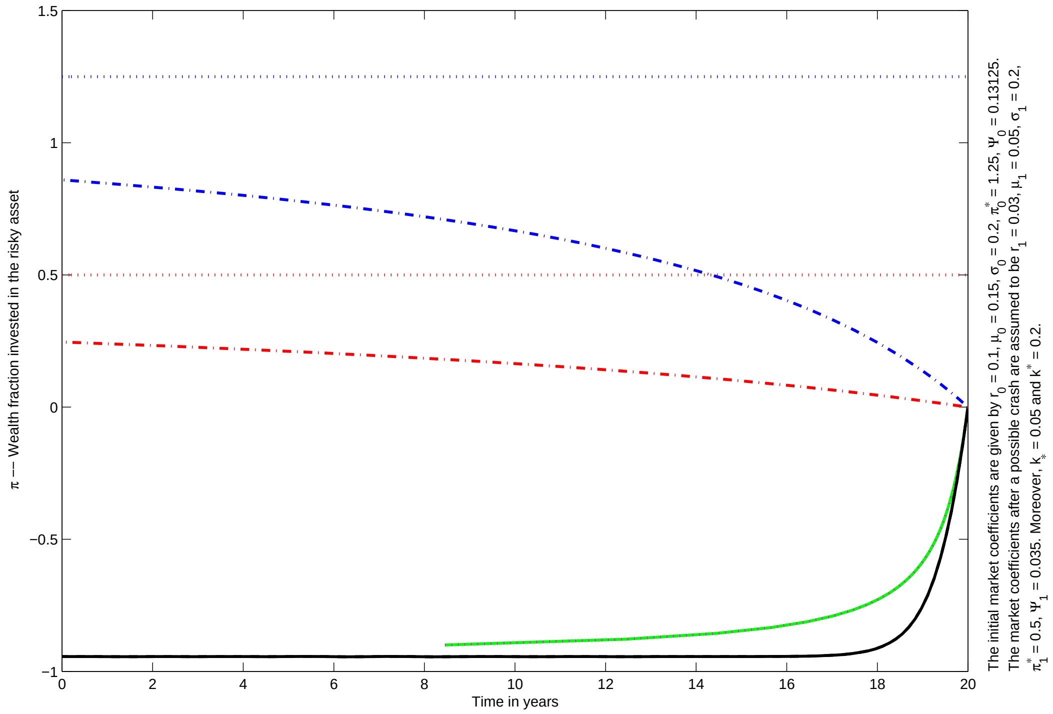
is negative. This means that the investor goes short of the risky asset in this situation. The same is true for the constant crash hedging strategy which is also a best constant worst case portfolio strategy (see Figure 7). This is due to the fact that the earning potential after a crash – Ψ_1 – is even less as the risk free interest rate today – r_0. This implies that the investor can increase his expected worst case utility by going short and thus taken sustantial losses into account as long as no crash happens. However, if a crash happens the investor is able to transfer some of his utility to the market 1 by using a short strategy which give him substantial gains in utility.

It is amazing that it is optimal for the investor to go short even if the probability of a crash is by no means sure.

As it has been stated in Proposition 2.15, the constant crash hedging strategy is equal to the best constant worst case portfolio strategy.

5. $\boldsymbol{\Psi_1 > \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* < 0}$
The crash hedging strategy in this case is positive and greater or equal than

Figure 7: Example $\Psi_1 < r_0$ and $\pi_0^* \geq 0$



This graphic shows $\hat{\pi} = \bar{\pi}$ (black line), $\bar{\varphi} = \hat{\varphi}$ (green line with cyan dotted points), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

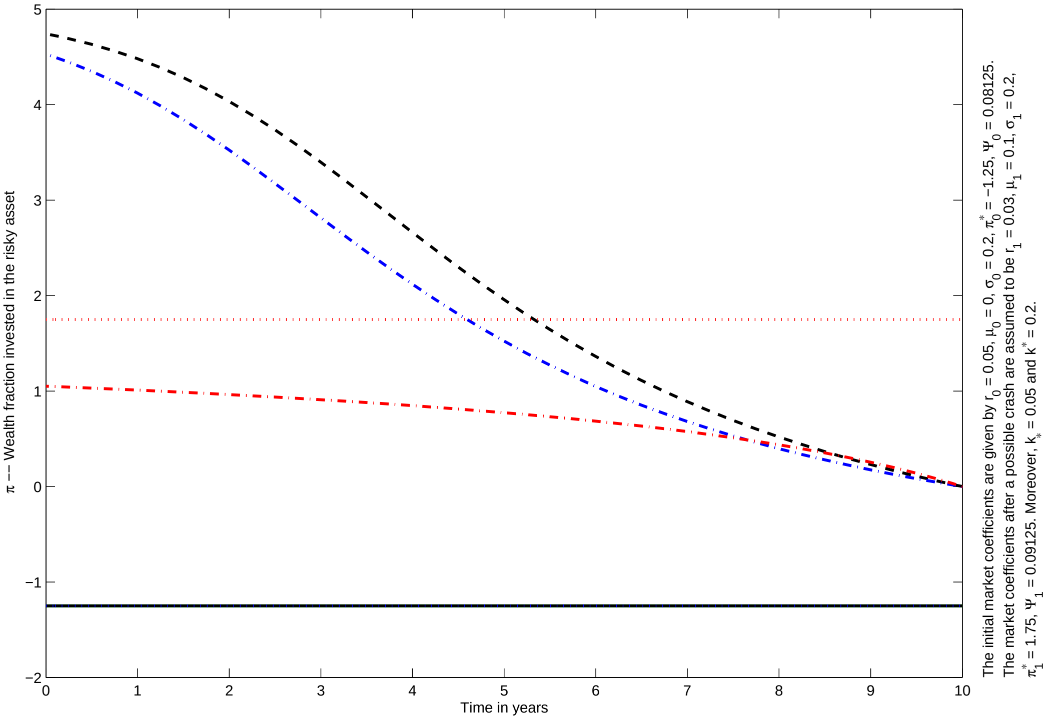
$\hat{\phi}_0$ (see Figure 8). This is even though $\hat{\phi}_1$ is only greater or equal than either the crash hedging strategy or $\hat{\phi}_0$ close to the terminal investment period. The optimal crash hedging strategy is given by π_0^*.

Furthermore, there exists no constant crash hedging strategy. However, the best constant worst case portfolio strategy is also given by π_0^*.

6. $\boldsymbol{r_0 \leq \Psi_1 \leq \Psi_0}$ ***and*** $\boldsymbol{\pi_0^* < 0}$
   In this situation the crash hedging strategy is positive (see Figure 9). Clearly, this is not optimal since it is like money (or utility for that reason) throwing away. Hence, it is optimal for the investor to take the portfolio strategy π_0^* at the end of her investment period and favoring a crash. However, if the investment period is so large that S defined in (17) is positive, the partial crash hedging strategy $\tilde{\pi}$, given the solution of (18) and (19), is an optimal crash hedging strategy and makes the investor crash indifferent on $[0, S]$. However, on $(S, T]$ a crash is favorable for the investor.

   Moreover, observe that $\hat{\varphi} = \bar{\varphi}$ on $[0, S]$. On $(S, T]$ no constant crash hedging strategy does exit. However, $\bar{\varphi} = \bar{\pi}$ on $(S, T]$, since $\bar{\pi}$ is constant on $(S, T]$.

Figure 8: Example $\Psi_1 > \Psi_0$ and $\pi_0^* < 0$



This graphic shows $\hat{\pi}$ (black dashed line), $\bar{\pi} = \bar{\varphi} = \pi_0^*$ (blue dotted line with black background), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), and π_1^* (red dotted line).

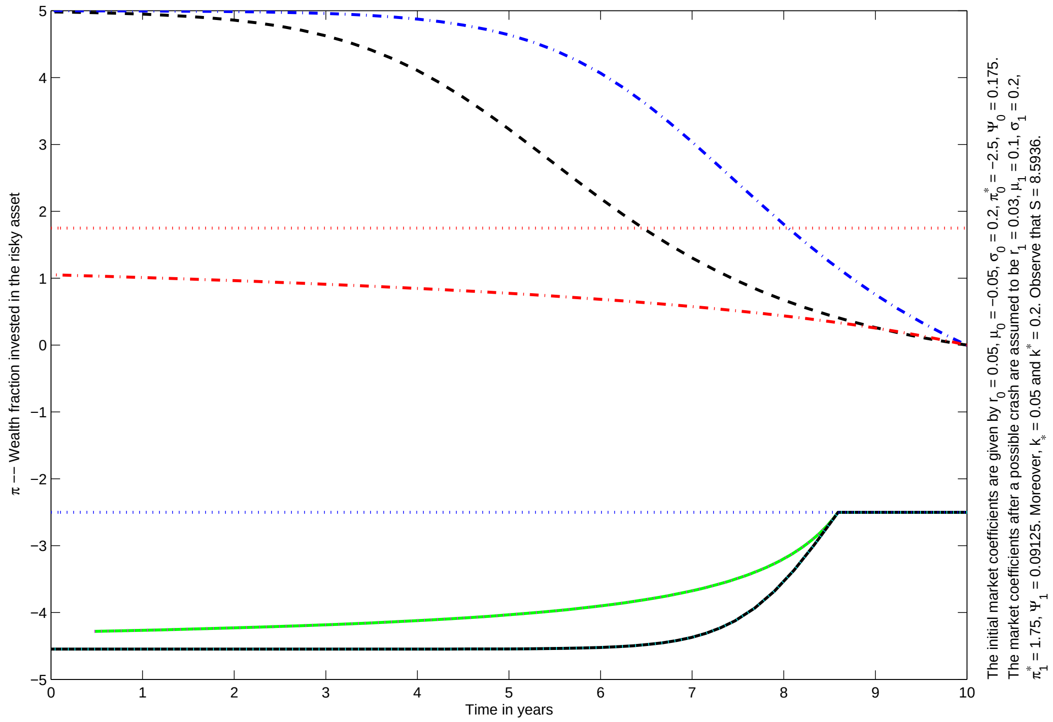
7. ***$\Psi_1 < r_0$ and $\pi_0^* < 0$***

The crash hedging strategy is negative (see Figure 10 and Figure 11). However, it is still not the optimal worst case portfolio strategy. As in the last case the optimal worst case porfolio strategy is given by $\tilde{\pi}$. For $S > 0$ the optimal crash hedging strategy is given by π_0^* on $[S, T]$ and by $\tilde{\pi}$ on $[0, S]$. Again, as in case 6, the investor is crash indifferent on $[0, S]$ and favors a crash on $(S, T]$ if she uses the optimal crash hedging stratety.

This case (as well as case 6) shows the *Bellman principle* or *optimality principle* (which is explained on page 5) nicely. Without knowing the Bellman principle, one might – *wrongly* – guess that $\min\{\hat{\pi}, \pi_0^*\}$ is the optimal crash hedging strategy. Since $\hat{\pi}$ is not optimal on $[S, T]$, it can neither be optimal on $[0, S]$, which is due to the Bellman principle. Therefore, applying the Bellman principle leads to the solution $\tilde{\pi}$.

Moreover, observe that $\hat{\varphi} = \bar{\varphi}$ on $[0, S]$. On $(S, T]$ no constant crash hedging strategy does exit. However, $\bar{\varphi} = \bar{\pi}$ on $(S, T]$, since $\bar{\pi}$ is constant on $(S, T]$.

Figure 9: Example $r_0 \leq \Psi_1 \leq \Psi_0$ and $\pi_0^* < 0$



This graphic shows $\hat{\pi}$ (black dashed line), $\bar{\pi} = \tilde{\pi}$ (black line), $\hat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

2.11 Hedging against a Regime Shift

Let us consider the case $k^* = k_* = 0$. This means that no crash is expected to happen. However, a so–called **regime shift** is supposed to occur, meaning that the market coefficients change.

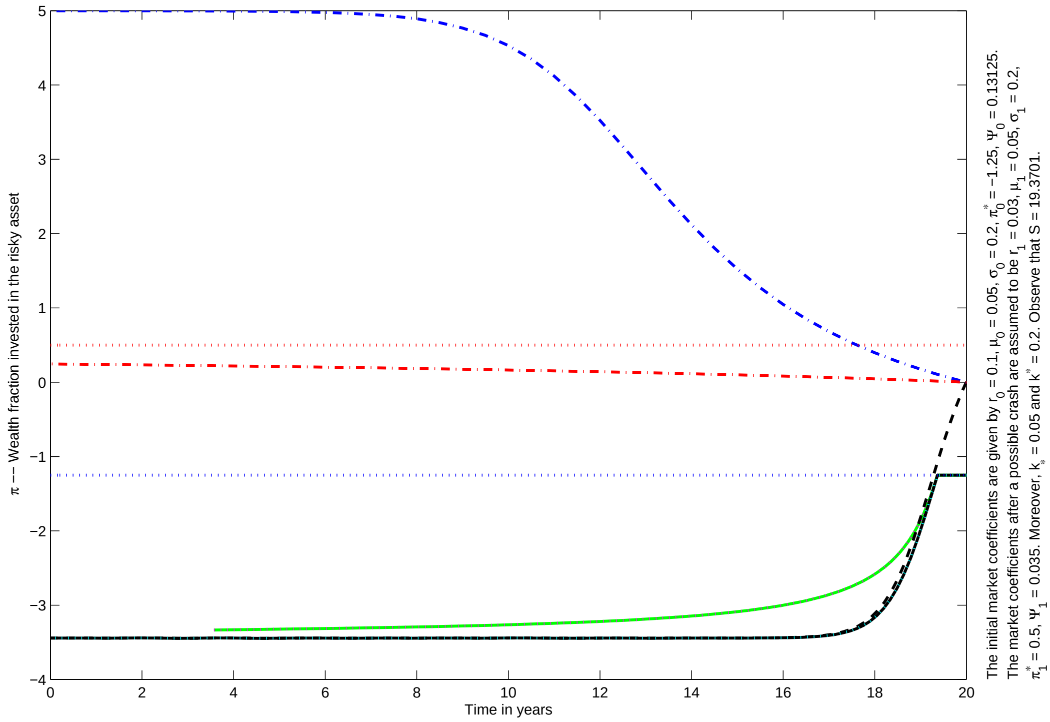
Thus, the former crash hedging equation (12) reduces to the following regime shift hedging equation.

$$0 = \int_t^T \left[\Psi_0 - \Psi_1 - \frac{\sigma_0^2}{2} \left(\hat{\pi}_{rs}(s) - \pi_0^* \right)^2 \right] ds,$$

where the hedging strategy against a regime shift is denoted by $\hat{\pi}_{rs}$. Analysing the above equation, it is straightforward to verify that the equation has only a solution if $\Psi_1 \leq \Psi_0$. Assuming this, it is easy to calculate that

$$\hat{\pi}_{rs}(s) = \pi_0^* \pm \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)}.$$

Figure 10: Example $\Psi_1 < r_0$ and $\pi_0^* < 0$, the long term behaviour



This graphic shows $\hat{\pi}$ (black dashed line), $\bar{\pi} = \tilde{\pi}$ (black line), $\hat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

Hence, it is possible to hedge against a regime shift only if $\Psi_1 \leq \Psi_0$. However, what does this strategy connote? The hedging strategy is to reduce the expected utility in the market 0 to the expected utility of market 1. Clearly, this strategy is not optimal. The optimal strategy is π_0^* – the classical optimal strategy in the model of Merton.

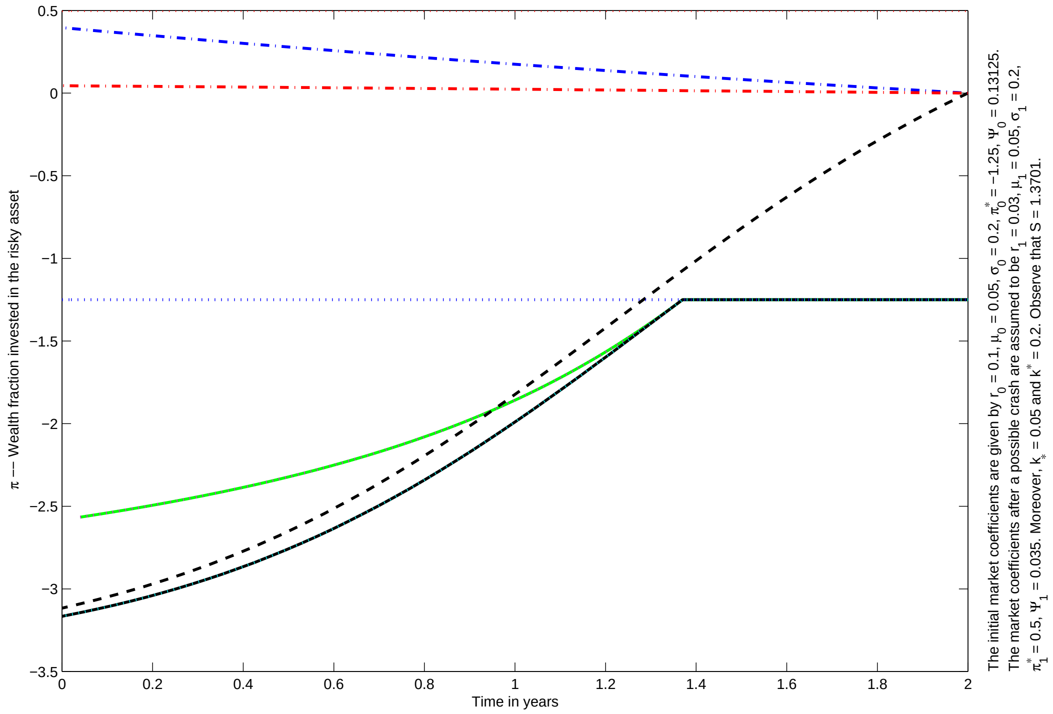
2.12 Implied Volatility

So far we have assumed that the time horizon T is the investment horizon. The assumption was then, that within this investment horizon, there exists the possibility of a crash.

Let us suppose now that the investment horizon is T and the time horizon for a possible crash is S_c with $S_c < T$. This means that the investor expects to see a crash in the time interval $[0, S_c]$. Thus, S_c will be called the possible ***crash horizon***. The smaller S_c is, the more imminent is a crash considered possible from the point of view of the investor.

Observe that the crash hedging strategy changes over time t, since the invest-

Figure 11: Example $\Psi_1 < r_0$ and $\pi_0^* < 0$



This graphic shows $\hat{\pi}$ (black dashed line), $\bar{\pi} = \tilde{\pi}$ (black line), $\hat{\varphi}$ (cyan dotted line), $\bar{\varphi}$ (green line), $\hat{\phi}_0$ (blue dash–dotted line), $\hat{\phi}_1$ (red dash–dotted line), π_0^* (blue dotted line), and π_1^* (red dotted line).

ment horizon changes over time. In other words, the crash hedging strategy $\hat{\pi}(t)$ belongs to the investment horizon $T - t$. Thus, if the investment horizon is $T + t$ for some $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ with $\varepsilon > 0$, the investment horizon would stay at T for that period. Correspondingly, the crash hedging strategy would be constantly staying at $\hat{\pi}\,(t_0)$ as long as the time is in the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$.

This observation justifies the following approach for the crash horizon. As long as the crash horizon S_c does not change and is smaller than the investment horizon T, the investor keeps a constant portfolio. More precisely, the investor keeps $\hat{\pi}(0; S_c)$ with an hypothetical investment horizon S_c.

$$\pi_c^*(S_c) := \hat{\pi}(0; S_c) := \hat{\pi}(0), \tag{49}$$

where $\hat{\pi}(0)$ is the initial investment strategy of an investor who has the investment horizon S_c. Using that the optimal portfolio process in the crash free model is given by π_0^*, equation (49) can be used to calculate an implied volatility.

$$\hat{\sigma}_c^2(S_c) := \frac{\mu_0 - r_0}{\hat{\pi}(0; S_c)}. \tag{50}$$

This is the implied variance in the crash model given a specified utility function and a crash horizon S_c. Subtracting the variance of the risky asset, one gets the implied variance (read risk) of a crash.

$$\sigma_c^2(S_c) := \left|\hat{\sigma}_c^2(S_c) - \sigma_0^2\right|.$$

Using the fact that

$$\hat{\pi}(0; S_c) = \hat{\pi}(T - S_c; T) \qquad \text{for arbitrary } T \geq S_c,$$

one can rewrite equation (50) as follows

$$\hat{\sigma}_c^2(S_c) := \frac{\mu_0 - r_0}{\hat{\pi}(T - S_c; T)} \qquad \text{for } T \geq S_c.$$

Therefore, it is possible to differentiate $\hat{\sigma}_c^2$ with respect to S_c

$$\frac{d\hat{\sigma}_c^2(S_c)}{dS_c} = \frac{\mu_0 - r_0}{\hat{\pi}^2(T - S_c; T)}\hat{\pi}'(T - S_c; T).$$

If π' is continuous from the right, one can take the limit $T \downarrow S_c$, yielding

$$\frac{d\hat{\sigma}_c^2(S_c)}{dS_c} = \frac{\mu_0 - r_0}{\hat{\pi}^2(0; S_c)}\hat{\pi}'(0; S_c),$$

thus showing the differentiability of the implied variance $\hat{\sigma}_c^2(S_c)$.

Using the differential equation (13), the derivative calculates to

$$\begin{aligned}
\frac{d\hat{\sigma}_c^2(S_c)}{dS_c} &= \frac{\mu_0 - r_0}{\hat{\pi}^2(0; S_c)}\hat{\pi}'(0; S_c) \\
&= \frac{\mu_0 - r_0}{\hat{\pi}^2(0; S_c)}\left(\hat{\pi}(0; S_c) - \frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}(0; S_c) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right].
\end{aligned}$$

The derivative of $\hat{\sigma}_c^2$ is decreasing for $\pi_0^* > 0$ and $\Psi_1 > r_0$, which is in accordance with intuition. The implied variance gets lower as the crash horizon gets farther away. The weaker threat of a possible crash is favorable for the investor. The same is true for $\pi_0^* < 0$ and $\Psi_1 < r_0$. Here, the threat of a crash comes not from the crash itself (since $\pi_0^* < 0$ it is acually favorable for the investor), but from the very bad earning potential after a crash $\Psi_1 < r_0$.

However, the derivative of $\hat{\sigma}_c^2$ is increasing for $\pi_0^* < 0$ and $\Psi_1 > r_0$. The implied variance gets greater as the crash horizon gets farther away. In this situation the investor would favor a crash happening. This is due to the fact that $\pi_0^* < 0$. The same is true for $\pi_0^* > 0$ and $\Psi_1 < r_0$. However, there is no explanation for this behavior.

Notice that it is also possible to define an implied volatility via the optimal crash hedging strategy $\bar{\pi}$ instead of the crash hedging strategy $\hat{\pi}$. Denoting this implied variance by $\bar{\sigma}_c^2$, it measures only the one–sided risk of a possible crash. $\bar{\sigma}_c^2$ does not measure both sides of a possible crash as $\hat{\sigma}_c^2$ does. Using $\bar{\sigma}_c^2$ the investor seems to be indifferent in certain situations to a change in the crash horizon which is clearly not the case.

This variance depends on the investors risk behavior as well as on the crash horizon, which is also fixed by the investor. Moreover, the implied variance depends on the market coefficient after a possible crash. Thus, this is only the individually perceived risk of a crash.

However, this model gives an explanation for the observed change in the volatility over time, which is not possible within the Black–Scholes model.

Furthermore, this calculated implicit volatility can be used for option pricing. Another application might be the possibility to calculate the intrinsic crash horizon of the market.

For another approach to value options in a jump diffusion model, see Merton [13]. There, however, one need the knowledge of the distribution of the jumps, while in the worst case scenario model one need to know only the crash horizon. However, both approaches make use of the utility function of the investor.

2.13 Optimal Portfolios Given the Probability of a Crash

Let us suppose in this section that the investor knows the probability of a crash occurring. Let p, with $p \in [0,1]$, be the probability of a crash occurring. The optimization problem writes to

$$\begin{aligned}
&\sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}_p\left[\ln\left(X^{\pi,t,x}(T)\right)\right] \\
&:= \sup_{\pi(\cdot)\in A(t,x)} \left\{ p\cdot\left\{ \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right]\right\} + (1-p)\,\mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right]\right\} \\
&= \sup_{\pi(\cdot)\in A(t,x)} \left\{ p\cdot\left\{ \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\nu_1\left(\tau, X_0^{\pi,t,x}(\tau)\left(1-\pi(\tau)k\right)\right)\right]\right\} + (1-p)\,\mathbb{E}\left[\nu_\pi\left(t,x\right)\right]\right\}.
\end{aligned}$$

Observe that the two extemes, $p \in \{0,1\}$ are straightforward to solve.

1. $p = 1$: $\displaystyle \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}_1\left[\ln\left(X^{\pi,t,x}(T)\right)\right] = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right].$

   Thus, this is the original worst case scenario portfolio problem. The solution is already known.

2. $p = 0$: $\displaystyle \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}_0\left[\ln\left(X^{\pi,t,x}(T)\right)\right] = \sup_{\pi(\cdot)\in A(t,x)} \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right],$

   which is the classical optimal portfolio problem. The solution is well–known and is given by π_0^*.

Let us now consider the case $p \in (0,1)$. Denoting the crash hedging strategy in this situation by $\hat{\pi}_p$ and the corresponding utility function by $\hat{\nu}_p\,(t,x) := \nu_{\hat{\pi}_p}\,(t,x)$, the defining equilibrium equation for the crash hedging strategy can be written as

$$\begin{array}{lrcl} & \hat{\nu}_p\,(t,x) & = & p\cdot\nu_1\,(t,x\,(1-\hat{\pi}_p(t)k^*)) + (1-p)\,\nu_{\hat{\pi}_p}(t,x) \\ \Longleftrightarrow & \hat{\nu}_p\,(t,x) & = & p\cdot\nu_1\,(t,x\,(1-\hat{\pi}_p(t)k^*)) + (1-p)\,\hat{\nu}_p(t,x) \\ \Longleftrightarrow & \hat{\nu}_p\,(t,x) & = & \nu_1\,(t,x\,(1-\hat{\pi}_p(t)k^*))\,, \end{array}$$

hence $\hat{\pi}_p \equiv \hat{\pi}$. This result shows that the crash hedging strategy remains the same even if the probability of a crash is known. Thus, this result justifies the wording *worst case szenario* of the above developed concept. This is due to the fact that the worst case szenario should be independent of the probability of the worst case and which has been shown above. Let us summarize this result in a proposition.

Proposition 2.28
The worst case scenario portfolio problem as it has been defined in Definition 2.2 is independent of the probability of the worst case, given that the probability of a crash is positive.

If the probability of a crash is zero, the worst case scenario portfolio problem reduces to the classical crash–free portfolio problem.

Obviously, the concept of the worst case szenario has the disadvantage that additional information (namely the given probability of a crash) is not used. However, if the probability of a crash and the probability of the crash size is known, it is possible to construct the **(lower) q–quantile crash hedging strategy**.

Assume that $p_c(t) \in [0,1]$ is the probability of a crash at time $t \in [0,T]$ and let $p(k,t) \in [0,1]$ be the density of the distribution function for a crash of size $k \in [k_*, k^*]$ at time t. Moreover, suppose that a function $q : [0,T] \longrightarrow [0,1]$ is given. With this define

$$k_q\,(t;\pi) := \left\{ \begin{array}{ll} 0 & \text{if } 1-p_c(t) \geq q(t) \\ \inf\left\{ k_q : 1-p_c(t)+p_c(t)\int\limits_{k_*}^{k_q} p(k,t)\,dk \geq q(t) \right\} & \begin{array}{l}\text{if } 1-p_c(t) < q(t) \\ \text{and } \pi \geq 0\end{array} \\ \sup\left\{ k_q : 1-p_c(t)+p_c(t)\int\limits_{k_q}^{k^*} p(k,t)\,dk \geq q(t) \right\} & \text{else} \end{array} \right\}$$

for any given portfolio strategy π. This has the following interpretation. The probability that at most a crash of size $k_q(t)$ at time t happens is $q(t)$. Equivalently, the probability that a crash higher than $k_q(t)$ will happen at time t is less than $1-q(t)$. Obviously, this is a value at risk approach.

Notice that the worst case of a nonnegative portfolio strategy is either a crash of size k^* or no crash. While the worst case of a negative portfolio strategy is either a crash of size k_* or no crash. Therefore, the q–quantile calculates differently for negative portfolio strategies (see the third row) than for the nonnegative portfolio strategies (see the second row). Furthermore, denote by

$$K_q(t) := \left\{ \begin{array}{ll} \{0\} & \text{if } k_q(t) = 0 \\ \{0\} \cup [k_*, k_q(t)] & \text{if } k_q(t) \neq 0 \text{ and } \pi \geq 0 \\ \{0\} \cup [k_q(t), k^*] & \text{else} \end{array} \right\}.$$

Definition 2.29

1. *The problem to solve*

$$\sup_{\pi(\cdot)\in A(x)} \inf_{\substack{0\leq\tau\leq T,\\ k\in K_q(t)}} \mathbb{E}\left[\ln\left(X^\pi(T)\right)\right], \tag{51}$$

where the final wealth $X^\pi(T)$ in the case of a crash of size k at time s is given by

$$X^\pi(T) = [1 - \pi(\tau)k]\, X_1^{\pi,\tau,X_0^\pi(\tau)}(T), \tag{52}$$

with $X_1^{\pi,\tau,X_0^\pi(\tau)}(t)$ as above, is called the **(lower) q–quantile scenario portfolio problem**.

2. *The* **value function** to the above problem *is defined via*

$$\nu_q(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\leq\tau\leq T,\\ k\in K_q(t)}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right]. \tag{53}$$

3. *A portfolio strategy $\hat{\pi}_q$ determined via the equation*

$$\nu_{\hat{\pi}_q}(t,x) = \nu_1\left(t, x\left(1 - \hat{\pi}_q(t)k_q(t)\right)\right) \qquad \text{for all } t \in [0,T] \text{ with } k_q(t) > 0$$

will be called a **(lower) q–quantile crash hedging strategy**.

4. *A portfolio strategy $\tilde{\pi}_q$ is a* **partial (lower) q–quantile crash hedging strategy**, *if it is for any $t \in [0,T]$ either a q-quantile crash hedging strategy or a solution to the q-quantile scenario portfolio problem.*

It is straightforward to see that the 1–quantile scenario portfolio problem is equivalent to the worst case scenario portfolio problem in defintion 2.2. Moreover, the 1–quantile crash hedging strategy is equivalent to the crash hedging strategy in definition 2.4.

Notice that the q–quantile scenario portfolio problem is only a q–quantile concerning the crash. The randomness of the market movement represented in the model by a Brownian Motion has been averaged out, namely by taking the expectation – and not the q–quantile.

Define the **support of k_q** to be

$$\operatorname{supp}(k_q) := \{t \in [0,T] : k_q(t) > 0\}.$$

Theorem 2.30
Let us suppose that k_q is continously differentiable on supp (k_q) with respect to t.

1. *Then there exists a unique (lower) q-quantile crash hedging strategy $\hat{\pi}_q$, which is on supp (k_q) given by the solution of the differential equation*

$$\begin{aligned}\hat{\pi}_q'(t) &= \left(\hat{\pi}_q(t) - \frac{1}{k_q(t)}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}_q(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right] - \hat{\pi}_q(t)k_q'(t), \quad (54)\\ \hat{\pi}_q(T) &= 0. \quad (55)\end{aligned}$$

   For $t \in [0,T] \setminus$ supp (k_q) set $\hat{\pi}_q(t) := \pi_0^*$.

   Moreover, the q–quantile crash hedging strategy is for $t \in$ supp (k_q) bounded by

$$0 \le \hat{\pi}_q(t) < \frac{1}{k_q(t)} \le \frac{1}{k_*} \quad \text{if } \Psi_1 \ge r_0.$$

   Additionally, if $\Psi_1 \le \Psi_0$ and $\pi_0^* \ge 0$, the q–quantile crash hedging strategy has another upper bound with $\hat{\pi}_q < \pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}$.

   On the other side, if $\Psi_1 < r_0$ the q–quantile crash hedging strategy is bounded by

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} < \hat{\pi}_q(t) < 0 \qquad \text{for } t \in [0,T).$$

2. *If $\Psi_1 < \Psi_0$ and $\pi_0^* < 0$, there exists a partial q–quantile crash hedging strategy $\tilde{\pi}_q$ at time t (which is different from $\hat{\pi}_q$), if*

$$S_q(t) := T - \frac{\ln\left(1 - \pi_0^* k_q(t)\right)}{\Psi_0 - \Psi_1} > 0 \qquad \text{for } t \in \text{supp}\,(k_q)\,. \quad (56)$$

   With this, $\tilde{\pi}_q(t)$ is given by the unique solution of the differential equation

$$\begin{aligned}\tilde{\pi}_q'(t) &= \left(\tilde{\pi}_q(t) - \frac{1}{k_q(t)}\right)\left[\frac{\sigma_0^2}{2}\left(\tilde{\pi}_q(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right] - \tilde{\pi}_q(t)k_q'(t),\\ \tilde{\pi}_q\left(S_q(t)\right) &= \pi_0^*.\end{aligned}$$

   For $S_q(t) \le 0$ set $\tilde{\pi}_q(t) := \pi_0^*$. This partial crash hedging strategy is bounded by

$$\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} < \tilde{\pi}_q \le \pi_0^* < 0.$$

If k_q is independent of the time t, the optimal portfolio strategy for an investor, who wants to maximize her q–quantile scenario portfolio problem, is given by

$$\bar{\pi}_q(t) := \min\left\{\hat{\pi}_q(t), \tilde{\pi}_q(t), \pi_0^*\right\} \qquad \text{for all } t \in [0,T], \quad (57)$$

where $\tilde{\pi}_q$ will be taken into account, if it exists. $\bar{\pi}_q$ will also be called the ***optimal q–quantile crash hedging strategy***.

Remark 2.31

1. It is also possible to solve the above problem if k_q is not continuously differentiable. In order to verify this define $\hat{\pi}_k$ to be the unique solution of

$$\begin{aligned} \hat{\pi}_k'(t) &= \left(\hat{\pi}_k(t) - \frac{1}{k}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}_k(t) - \pi_0^*\right)^2 + \Psi_1 - \Psi_0\right] \quad \text{and} \quad (58) \\ \hat{\pi}_k(T) &= 0, \quad (59) \end{aligned}$$

for $k > 0$. Set then $\hat{\pi}_q(t) := \hat{\pi}_{k_q(t)}(t)$ where the convention $\hat{\pi}_0(t) := \pi_0^*$ is used in order to include the case $k_q(t) = 0$. Note that this procedure is also possible for continuously differentiable k_q. However, only if k_q is continuously differentiable, it is possible that $\hat{\pi}_q$ is also continuously differentiable.

2. Notice that $\hat{\pi}_{k_1}' < \hat{\pi}_{k_2}'$ for $k_1 < k_2$. Hence, $\hat{\pi}_{k_1} \geq \hat{\pi}_{k_2}$ with strict inequality applying on $[0, T)$. Thus, in particular, $\hat{\pi}_q(t) > \hat{\pi}(t)$ for $t \in [0, T)$ for any q which satisfies $q(t) < 1$ for $t \in [0, T)$. Moreover, $\hat{\pi}_{q_1}(t) \leq \hat{\pi}_{q_2}(t)$, if $q_1 > q_2$. Hence, any $\hat{\pi}_q$ will be in area ii) of Figure 1 (and possibly also in area iii) and iv), if $\Psi_1 > \Psi_0$).

3. For this remark, let us suppose that the market conditions do not change, hence $\Psi_1 = \Psi_0$. Moreover, keep in mind that any $\hat{\pi}_k$ is bounded by π_0^* from above. Thus, it is clear that

$$\psi(t) := \left\{ \begin{array}{ll} 0 & \text{for } t = T \\ \pi_0^* & \text{else} \end{array} \right\}$$

is an upper bound for any $\hat{\pi}_k$ with $k > 0$. Unfortunately, it is not possible to show that

$$\hat{\pi}_{k^*} \longrightarrow \psi$$

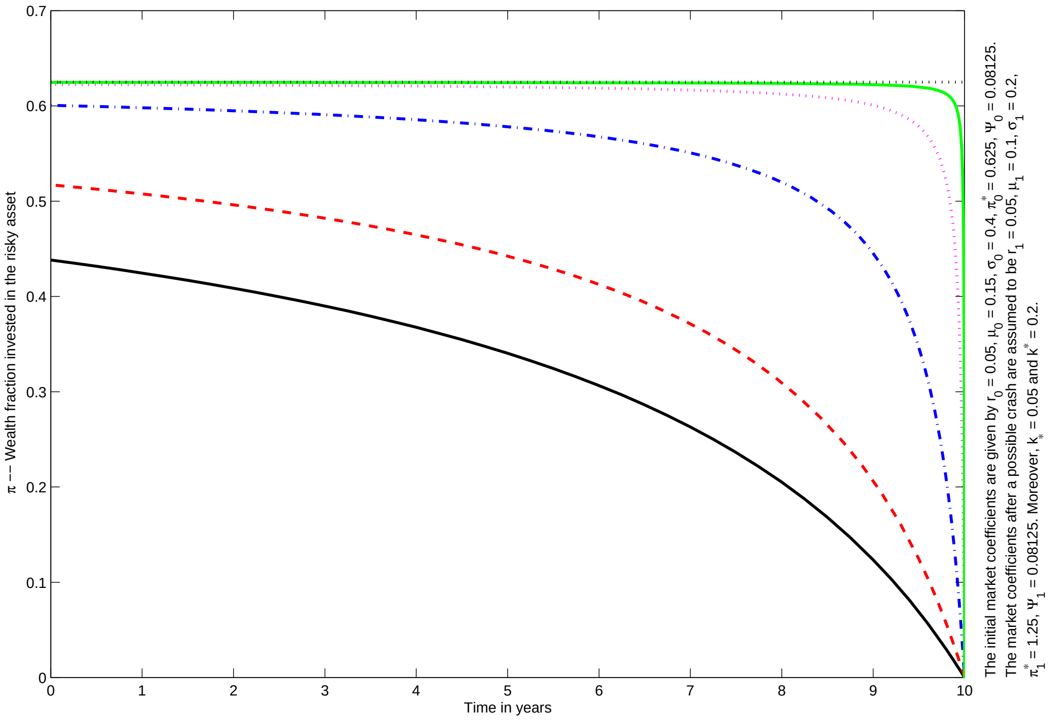
for $k^* \downarrow 0$ with $k^* \neq 0$, since $\hat{\pi}_{k^*}$ is only known implicitly and not explicitly. However, this is exactly what can be observed in practise (see e.g. Figure 12).

Moreover, keep in mind that the case $k = 0$ yields π_0^* as the optimal portfolio with $\pi_0^* \not\equiv \psi$.

Proof of Theorem 2.30: If $k_q(t)$ is constant in t this theorem follows from Theorem 2.5 by replacing k^* with k_q. To verify the differential equation in the general case, keep in mind that by differentiating the – modified – equation (12) with respect to t, $k_q(t)$ has also to be differentiated with respect to t. This leads to the differential equation (54). □

The following figures (see Figure 13 and Figure 14) show the potential range of the q–quantile crash hedging strategy and the optimal q–quantile crash hedging strategy if $k_q(t) \neq 0$. In the case of $k_q(t) = 0$, $\hat{\pi}_q(t) = \pi_0^*$ as well as $\bar{\pi}_q(t) = \pi_0^*$.

Figure 12: Example of $k \longrightarrow 0$ for $\Psi_1 = \Psi_0$ and $\pi_0^* \geq 0$



This graphic shows $\hat{\pi} = \hat{\pi}_{k^*}$ (black dashed line), $\hat{\pi}_{\frac{k^*}{2}}$ (red dashed line), $\hat{\pi}_{\frac{k^*}{10}}$ (blue dash–dotted line), $\hat{\pi}_{\frac{k^*}{100}}$ (cyan dotted line), $\hat{\pi}_{\frac{k^*}{1000}}$ (green solid line), and π_0^* (black dotted line).

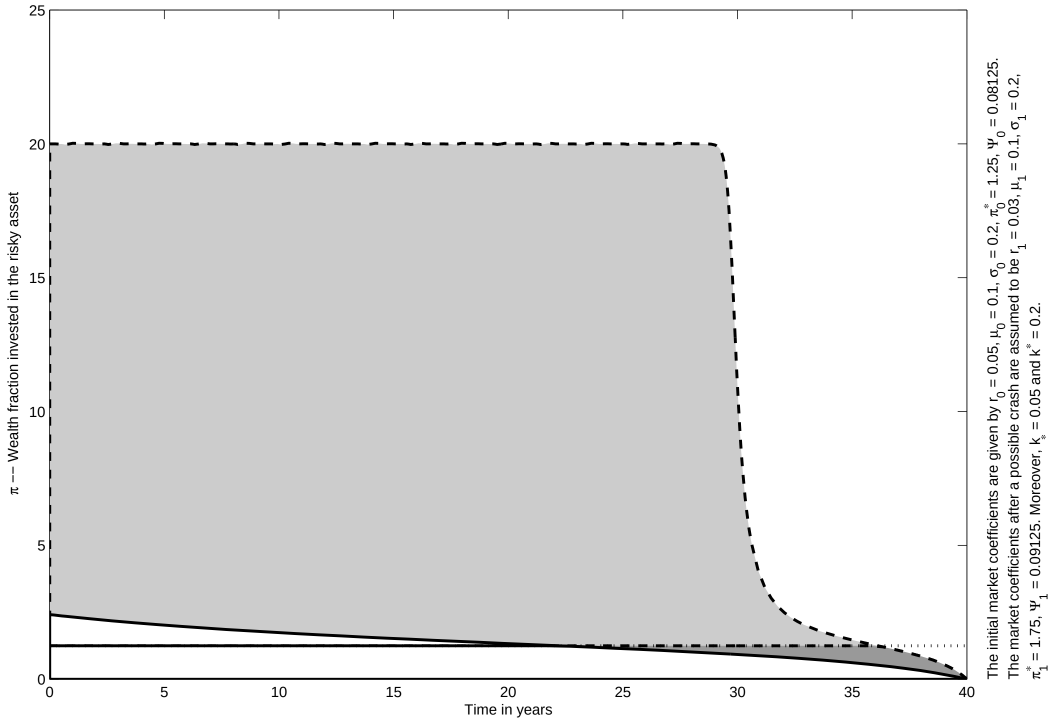
Figure 13 is an example of the case $\Psi_1 > \Psi_0$ and $\pi_0^* \geq 0$. The market coefficients are the same as in Figure 4, only the investment horizon is different. Notice that $\hat{\pi}_{k_*}$ comes way faster close to its boundary $\frac{1}{k_*}$ (which is in this example equal to 20) than $\hat{\pi}_{k^*}$ to its boundary $\frac{1}{k^*}$ (which is in this example equal to 5).

Figure 14 is an example of the case $r_0 < \Psi_1 < \Psi_0$ and $\pi_0^* < 0$. The market coefficients are the same as in Figure 9. For the q–quantile crash hedging strategies the same statements hold as in the first example. However, the optimal q–quantile crash hedging strategies are given by the partial q–quantile crash hedging strategy $\tilde{\pi}_q$. Note that $\tilde{\pi}_q$ depends on S_q which itself depends on k_q. Observe that S (which corresponds to k_*) is greater than S^* (which corresponds to k^*).

2.14 Comparison with Traditional Crash Modelling

In the standard theory of portfolio optimization the possibility of a crash is taken into account by considering a diffusion with jumps instead of a diffusion (see for example Merton [13] or Aase [1]). In order to keep things simple, let us assume

Figure 13: The Range of (Optimal) q–Quantile Crash Hedging Strategies for $\Psi_1 > \Psi_0$ and $\pi_0^* \geq 0$



This graphic shows $\hat{\pi}_{k^*}$ (black solid line), $\hat{\pi}_{k_*}$ (black dashed line), the range of possible q–quantile crash hedging strategies (light grey and dark grey area), the range of possible optimal q–quantile crash hedging strategies (dark grey area), and π_0^* (black dotted line).

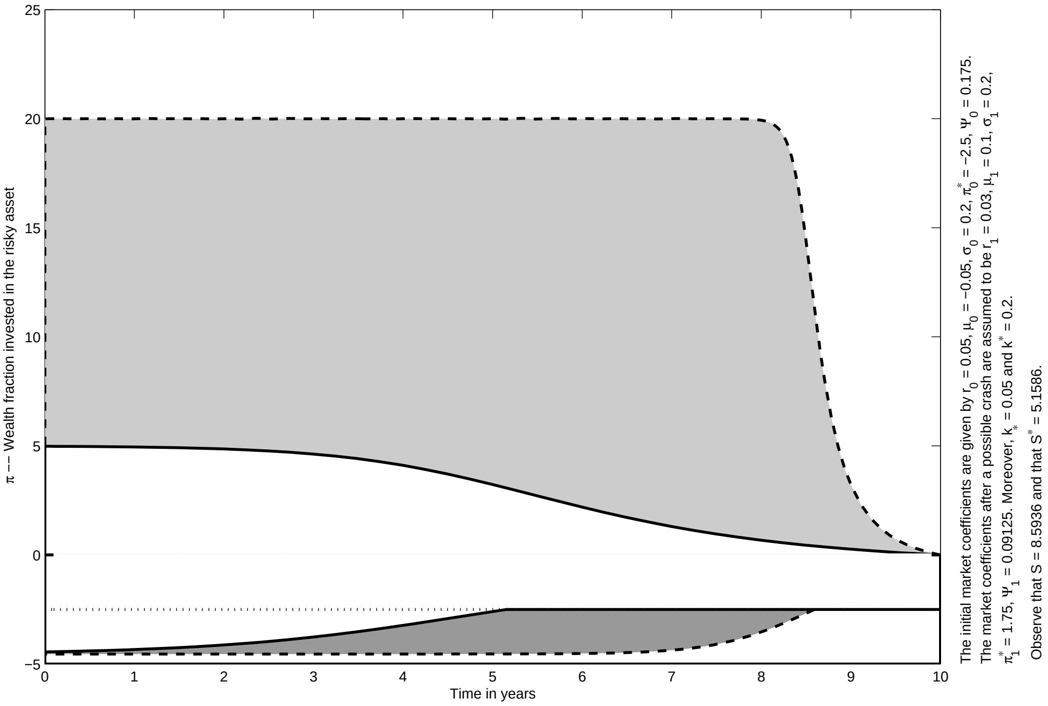
that the jump process is a summation of m different Poisson processes N_j with intensity λ_j and jump size k_j (for $j = 1, \ldots, m$). With this the price process of the risky asset writes to

$$dP_{0,1}(t) = P_{0,1}(t) \left[\mu_0 \, dt + \sigma_0 \, dW(t) - \sum_{j=1}^{m} k_j \, dN_j(t) \right], \qquad P_{0,1}(0) = p_1 \,. \quad (60)$$

Given log–utility, the value function for a portfolio strategy π calculates to

$$\begin{aligned} \nu_{P,\pi}(t,x) &= \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T) \right) \right] \\ &= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2} \left(\pi(s) - \pi_0^* \right)^2 \right. \right. \\ &\qquad \left. \left. + \sum_{j=1}^{m} \ln\left(1 - \pi(s) k_j \right) \lambda_j(s) \right] ds \right]. \end{aligned}$$

Figure 14: The Range of (Optimal) q–Quantile Crash Hedging Strategies for $r_0 < \Psi_1 < \Psi_0$ and $\pi_0^* < 0$



This graphic shows $\hat{\pi}_{k^*}$ and $\bar{\pi}_{k^*}$ (black solid line), $\hat{\pi}_{k_*}$ and $\bar{\pi}_{k_*}$ (black dashed line), the range of possible q–quantile crash hedging strategies (light grey area), the range of possible optimal q–quantile crash hedging strategies (dark grey area), and π_0^* (black dotted line).

If π as well as λ_j are deterministic for $j = 1, \ldots, m$, the expectation is redundant.

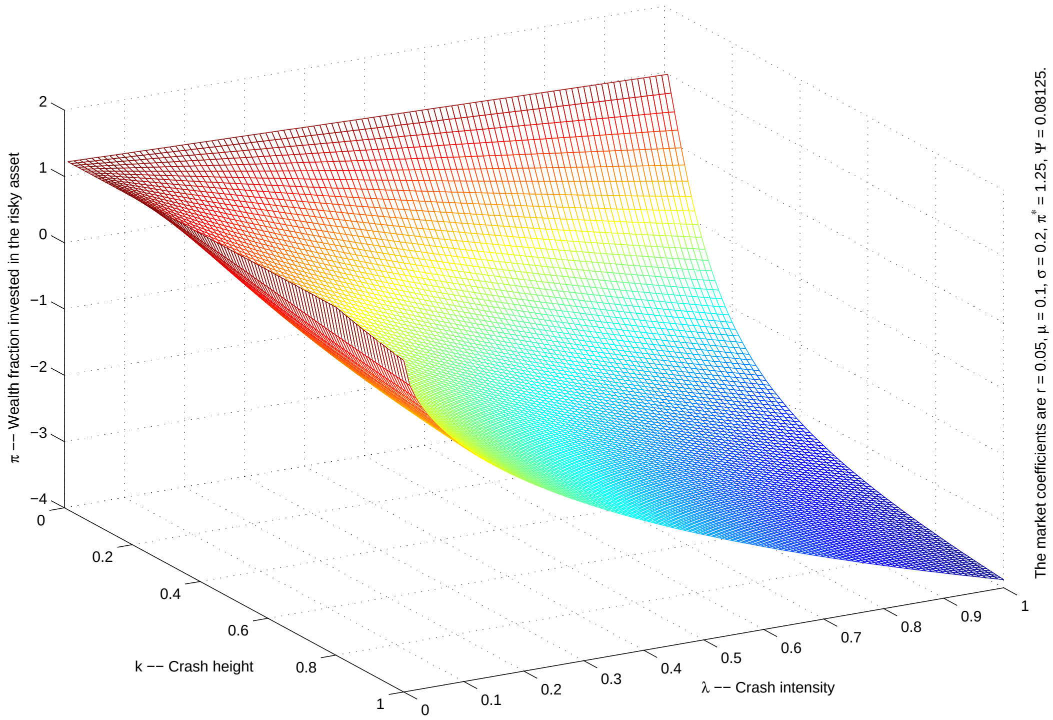
Assuming that $m = 1$, $k_1 = k$ and $\lambda_1 = \lambda$ is deterministic, the optimal portfolio strategy π_P^* calculates to

$$\begin{aligned}\pi_P^*(t) &= \frac{\sigma_0^2 + k\,(\mu_0 - r_0)}{2k\sigma_0^2} - \sqrt{\left(\frac{\sigma_0^2 + k\,(\mu_0 - r_0)}{2k\sigma_0^2}\right)^2 + \frac{k\lambda(t) - (\mu_0 - r_0)}{k\sigma_0^2}} \\ &= \frac{1}{2}\left(\pi_0^* + \frac{1}{k}\right) - \sqrt{\frac{1}{4}\left(\pi_0^* - \frac{1}{k}\right)^2 + \frac{\lambda}{\sigma_0^2}}\,.\end{aligned}$$

For the derivation of the value function $\nu_{P,\pi}$ and the optimal portfolio strategy π_P^* see the Appendix. Figure 15 shows how the optimal portfolio strategy π_P^* depends on the crash height and the crash intensity for some specific market coefficients.

Figure 16 depicts the level lines of the optimal portfolio strategy π_P^* over the crash height and the probability that no crash occurs within the next year. Given

Figure 15: The Dependence of the Optimal Portfolio Strategie π_P^* on the Crash Height and the Crash Intensity



This graphic shows the dependence of π_P^* from the crash height k and the crash intensity λ. The market coefficients are assumed to be $r_0 = 0.05$, $\mu_0 = 0.1$ and $\sigma_0 = 0.2$.

a crash height of $k = 0.2$, note that the optimal portfolio strategy π_P^* is in this setting already negative if the probability of no crash occurring is about 0.7 or less.

Clearly,

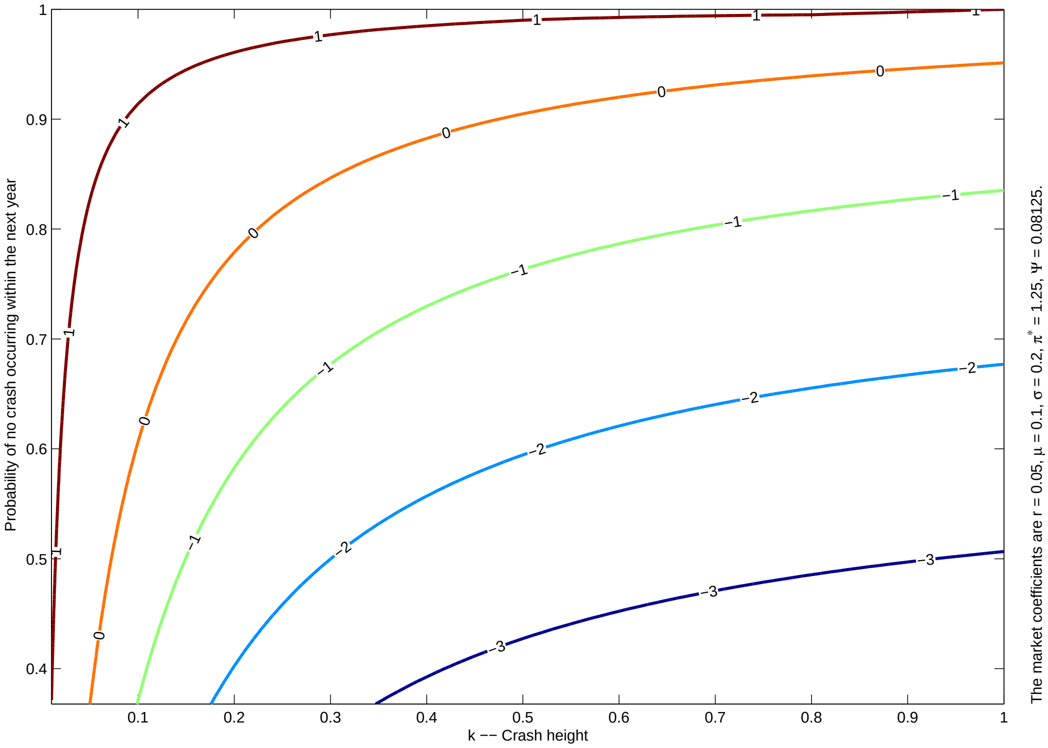
$$\pi_P^*(t) \begin{cases} > 0 & \text{for } \lambda(t) < \frac{\mu_0 - r_0}{k}, \text{ given that } \mu_0 > r_0 \\ = 0 & \text{for } \lambda(t) = \frac{\mu_0 - r_0}{k}, \text{ given that } \mu_0 > r_0 \\ < 0 & \text{else} \end{cases} \Bigg\} .$$

In accordance with the **market price of risk** (which is defined as $\frac{\mu_0 - r_0}{\sigma}$), $\frac{\mu_0 - r_0}{k}$ will be called the **market price of a crash with size k**. For $t \longrightarrow T$, one has

$$\begin{array}{rcl} & \pi_P^*(t) & \longrightarrow & 0 \\ \Longleftrightarrow & \lambda(t) & \longrightarrow & \dfrac{\mu_0 - r_0}{k}. \end{array}$$

Observe that it is possible that the optimal crash hedging strategy $\bar{\pi}$ is positive and the optimal portfolio strategy π_P^* is negative. This is due to the different

Figure 16: Level Lines of the Optimal Portfolio Strategie π_P^*



This graphic shows the level lines of the optimal portfolio strategy π_P^*. The variables are the crash height k and the probability that no crash occurs within the next year. The market coefficients are assumed to be $r_0 = 0.05$, $\mu_0 = 0.1$ and $\sigma_0 = 0.2$.

optimization problems. However, this does not imply that the expected utility of the investor is in the worst case portfolio problem greater than in the above portfolio problem model for price processes with jumps. Keeping the above example (see Figure 16) in mind, there might be the "feeling" that the optimal portfolio strategy π_P^* is in some cases too risky in some sense.

The worst case scenario portfolio problem approach is an alternative which remedies some aspects of the classical portfolio problem. The main aspects are that

- there are no probabilistic assumptions about the crash,
- the wealth fraction invested in the risky asset will be reduced to zero as the terminal investment time approaches (if $\pi_0^* \geq 0$), and
- the optimal crash hedging strategy does not "feel" as risky as the optimal portfolio strategy π_P^*.

However, it remains to find some mathematical function or procedure which can quantitize the feeling of one strategy being riskier than another.

3 Considering General Utility Functions

3.1 The Set Up

As in the case of logarithmic utility, let us start with the most basic setting and consider a security market consisting of a riskless bond and a single risky security with prices given by

$$\begin{aligned} dP_{0,0}(t) &= P_{0,0}(t)\, r_0\, dt\,, & P_{0,0}(0) = 1\,, \qquad (61)\\ dP_{0,1}(t) &= P_{0,1}(t)\left[\mu_0\, dt + \sigma_0\, dW(t)\right]\,, & P_{0,1}(0) = p_1\,, \qquad (62) \end{aligned}$$

with constant market coefficients μ_0, r_0 and $\sigma_0 \neq 0$ in "normal times" and where W is a Brownian Motion on a complete probability space $(\Omega, \mathcal{F}, P)$. Assume further that at most one crash can happen within the time horizon T. At the "crash time" the stock price suddenly falls. More specific, suppose that the sudden relative fall of the stock price lies in the interval $[k_*, k^*]$, where the constants $0 < k_* < k^* < 1$ ("the lowest and the worst possible crash size, respectively") are given.

Again, let us model the occurrence of a crash by a jump process $N(t)$ which is zero before the jump time and equals one from the jump time onwards. Let us require that N lives also on $(\Omega, \mathcal{F}, P)$. To model the fact that the investor is able to realize that a jump of the stock price has happened it is supposed that the investor's decisions are adapted to the P-augmentation $\{\mathcal{F}_t\}$ of the filtration generated by both the Brownian motion $W(t)$ and the jump process $N(t)$.

Let us further suppose that the market conditions change after a possible crash. Moreover suppose that the change in the coefficients depends on the height of the crash. Thus, after a crash of height k, we observe the coefficients $r_1(k)$, $\mu_1(k)$, and $\sigma_1(k) \neq 0$. Let therefore k (with $k \in [k_*, k^*]$) be the arbitrary size of a crash at time τ. The price of the bond and the risky asset is assumed to be

$$\begin{aligned} dP_{1,0}(t;k) &= P_{1,0}(t;k)\, r_1(k)\, dt\,, & P_{1,0}(\tau;k) = P_{0,0}(\tau)\,, \qquad (63)\\ dP_{1,1}(t;k) &= P_{1,1}(t;k)\left[\mu_1(k)\, dt + \sigma_1(k)\, dW(t)\right], & P_{1,1}(\tau;k) = (1-k)\, P_{0,1}(\tau), \qquad (64) \end{aligned}$$

after a possible crash of size k at time τ.

For simplicity, the initial market will also be called market 0, while the market after a crash will be called market 1.

In order to avoid redundant definitions, the following definitions will be given for market i with $i \in \{0, 1\}$ and for crash height k with $k \in [k_*, k^*]$. However, market 0 is independent of k. Hence, the variable k is redundant if the definitons are applied for $i = 0$. This fact has to be kept in mind for the following definitions.

Definition 3.1

1. *For $i \in \{0,1\}$, let $\boldsymbol{A_i(s, x; k)}$ be the* **set of admissible portfolio processes** *$\pi(t)$ corresponding to an initial capital of $x > 0$ and a crash height of k at time s, i.e. $\{\mathcal{F}_t, s \le t \le T\}$–progressively measurable processes such that*

(i) *the* **wealth equation** *in market i in the usual crash–free setting*

$$\begin{aligned} dX_i^{\pi,s,x}(t;k) &= X_i^{\pi,s,x}(t;k)\left[(r_i(k)+\pi(t)\left[\mu_i(k)-r_i(k)\right]\right)dt \qquad (65)\\ &\quad + \pi(t)\sigma_i(k)\,dW_i(t)\right], \\ X_i^{\pi,s,x}(s;k) &= x \qquad (66)\end{aligned}$$

has a unique non–negative solution $X_i^{\pi,s,x}(t;k)$ and satisfies

$$\int_s^T \left[\pi(t)X_i^{\pi,s,x}(t;k)\right]^2 dt < \infty \quad P\text{–a.s.}, \qquad (67)$$

i.e. $\boldsymbol{X_i^{\pi,s,x}(t;k)}$ *is the* **wealth process in market** $\boldsymbol{i}$ *in the crash–free world, which uses the portfolio strategy π and starts at time s with initial wealth x.*

Furthermore, $X_i^{\pi}(t;k) := X_i^{\pi,0,x}(t;k)$ will be used as an abbreviation.

(ii) *$\pi(t)$ has left–continuous paths with right limits.*

2. *the corresponding* **wealth process $\boldsymbol{X^\pi(t)}$ in the crash model**, *defined as*

$$X^\pi(t) = \begin{cases} X_0^\pi(t) & \text{for } s \le t < \tau \\ \left[1-\pi(\tau)k\right] X_1^{\pi,\tau,X_0^\pi(\tau)}(t;k) & \text{for } t \ge \tau \ge s\,, \end{cases} \qquad (68)$$

given the occurrence of a jump of height k at time τ, is strictly positive. Thereby, it is assumed that the crash time τ is a stopping time, which is supposed to be $\mathcal{F}_t$–measurable. The set of admissible portfolio strategies is obviously given by $A_0(s,x)$ as long as no crash happens. After a crash of height k at time τ the set is given by $A_1(\tau,x;k)$. Hence,

$$A(s,x) := A_0(s,x)\Big|_{[0,\tau]} \cup A_1(\tau,x;k).$$

3. *The market model X_1 is called* **consistent** *with the market model X_0, if $X_1^{\pi,x,s}(t;0) = X_0^\pi(t)$. This is a natural definition, since $k=0$ means, that there is no crash so far. Thus, in $k=0$ the market model X_0 rules.*

4. *$A(x)$ is used as an abbreviation for $A(0,x)$.*

With these definitions it is possible to state the worst case scenario portfolio problem. Note that due to the lack of statistical assumptions on the distribution of both the crash height and the crash time, the problem cannot be dealt with by simply maximizing the expected utility of final wealth. However, the crash consequence has to be taken into account in some way. The approach of this thesis is to maximize the utility of the worst case possible.

Let U_i be the utility function (i.e. a strictly concave, monotonously increasing and differentiable function) of the investor being in market i for $i \in \{0,1\}$. This can be interpreted in such a way that the investor values the risk differently after she actually observed a crash. Given that a crash happens at time s with $s \in [0,T]$, denote

$$U_s(x) := U_0(x) \cdot \mathbb{1}_{[0,s)} + U_1(x) \cdot \mathbb{1}_{[s,T]}.$$

With this convention it is possible to make the following definition.

Definition 3.2

1. *The problem to solve*

$$\sup_{\pi(\cdot)\in A(x)} \inf_{\substack{0\le\tau\le T,\\ k\in K}} \mathbb{E}\left[U_\tau\left(X^\pi(T)\right)\right] , \tag{69}$$

   where the final wealth $X^\pi(T)$ in the case of a crash of size k at time τ is given by

$$X^\pi(T) = [1-\pi(\tau)k]\, X_1^{\pi,\tau,X_0^\pi(\tau)}(T;k) , \tag{70}$$

   with $X_1^{\pi,\tau,X_0^\pi(\tau)}(t;k)$ as above, is called the **worst case scenario portfolio problem**.

2. *The* **value function** to the above problem *is defined via*

$$\nu_c(t,x) = \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\le\tau\le T,\\ k\in K}} \mathbb{E}\left[U_\tau\left(X^{\pi,t,x}(T)\right)\right] . \tag{71}$$

3. *The* **value function** in the crash–free setting **of market $\boldsymbol{i}$** *will be denoted*

$$\nu_i(t,x;k) = \sup_{\pi(\cdot)\in A_i(t,x;k)} \mathbb{E}\left[U_i\left(X_i^{\pi,t,x}(T;k)\right)\right] .$$

4. *The* **value function** in the market i **given the admissible portfolio strategy $\boldsymbol{\pi}$** *will be named*

$$\nu_{i;\pi}(t,x;k) = \mathbb{E}\left[U_i\left(X_i^{\pi,t,x}(T;k)\right)\right] .$$

   Furthermore, $\nu_\pi(t,x) := \nu_{0;\pi}(t,x)$.

In the sequel the following assumptions are needed.

Assumption 3.3

A.1 Let us suppose that the value functions $\nu_i(t,x;k)$ (for $i \in \{0,1\}$) are continuously differentiable in t, twice continuously differentiable in x, and twice continuously differentiable in k. This fact is expressed by saying that $\nu_i(t,x;k)$ is a $C^{1,2,2}$–function.

A.2 Let $\nu_i(t,x;k)$ be strictly increasing and strictly concave in x for $i \in \{0,1\}$.

In order to get shorter and more transparent formulae, the following definitions are useful.

Definition 3.4
Suppose that assumption A.1 holds. Then define for $i, j \in \{0, 1\}$

1. *the* **generalized optimal portfolio strategy** in market i given the value function of market j, *assuming that no crash will happen, by*

$$\pi^*_{ij}(t, x; k_i, k_j) := -\frac{\mu_i(k_i) - r_i(k_i)}{\sigma_i^2(k_i)} \frac{(\nu_j)_x (t, x; k_j)}{x (\nu_j)_{xx} (t, x; k_j)}.$$

2. *Moreover,*

$$\begin{aligned}\Psi_{ij}(t, x; k_i, k_j) &:= r_i(k_i) - \frac{1}{2} \frac{(\nu_j)_x (t, x; k_j)}{x (\nu_j)_{xx} (t, x; k_j)} \left(\frac{\mu_i(k_i) - r_i(k_i)}{\sigma_i(k_i)} \right)^2 \\ &= r_i(k_i) - \frac{\sigma_i^2(k_i)}{2} \frac{x (\nu_j)_{xx} (t, x; k_j)}{(\nu_j)_x (t, x; k_j)} \left(\pi^*_{ij}(t, x; k_i, k_j) \right)^2\end{aligned}$$

will be called the **generalized earning potential** of market i given the value function of market j.

If market i has not been entered by a crash or if the crash size is irrelevant for market i (as it is the case for market 0), the notation $\pi^*_{ij}(t, x; k)$ and $\Psi_{ij}(t, x; k)$ will be used instead of $\pi^*_{ij}(t, x; k_i, k_j)$ and $\Psi_{ij}(t, x; k_i, k_j)$, respectively. Furthermore, denote

$$\begin{aligned}\pi^*_i(t, x; k) &:= \pi^*_{ii}(t, x; k) \qquad \text{and} \\ \Psi_i(t, x; k) &:= \Psi_{ii}(t, x; k) = -\frac{(\nu_i)_t (t, x; k)}{x (\nu_i)_x (t, x; k)},\end{aligned}$$

where the last equation is due to the HJB–equation for ν_i: Assuming that A.1 holds, one has

$$\begin{aligned}(\nu_i)_t (t, x; k) + x (\nu_i)_x (t, x; k) \left[r_i(k) + \pi^*_i(t, x; k) (\mu_i(k) - r_i(k)) \right] & \\ + \frac{1}{2} x^2 (\nu_i)_{xx} (t, x; k) \sigma_i^2(k) \left(\pi^*_i(t, x; k) \right)^2 &= 0.\end{aligned}$$

Using the definition of π^*_i which can be rewritten as

$$-(\mu_i(k) - r_i(k)) = \frac{x (\nu_i)_{xx} (t, x; k)}{(\nu_i)_x (t, x; k)} \sigma_i^2(k) \pi^*_i(t, x; k),$$

the HJB–equation of the value function leads to the following equation.

$$-\frac{(\nu_i)_t (t, x; k)}{x (\nu_i)_x (t, x; k)} = r_i(k) + \pi^*_i(t, x; k) (\mu_i(k) - r_i(k))$$

$$
\begin{aligned}
& + \frac{\sigma_i^2(k)}{2} \frac{x\,(\nu_i)_{xx}\,(t,x;k)}{(\nu_i)_x\,(t,x;k)} \left(\pi_i^*\,(t,x;k)\right)^2 \\
= \; & r_i(k) - \frac{\sigma_i^2(k)}{2} \frac{x\,(\nu_i)_{xx}\,(t,x;k)}{(\nu_i)_x\,(t,x;k)} \left(\pi_i^*\,(t,x;k)\right)^2 ,
\end{aligned}
$$

which shows the stated assertion.

Remark 3.5

1. The ***utility growth potential*** is in this situation given by $-\,(\nu_i)_t\,(t,x;k)$. Obviously, if $x\,(\nu_i)_x\,(t,x;k) = 1$ the *utility growth potential* is equivalent to the *earning potential* as in the case of logarithmic utility. The minus sign is due to the fact that ν_i is derived by a backward equation.

2. If $U_j = U_i$ and if $\frac{x(\nu_j)_{xx}(t,x;k)}{(\nu_j)_x(t,x;k)}$ is independent of the market conditions of market j, one has $\pi_{ij}^* = \pi_i^*$ and $\Psi_{ij} = \Psi_i$ for all i and j. For instance, this is the case for the logarithmic utility or for the HARA–utility functions.

3. If $\frac{x(\nu_j)_{xx}(t,x;k)}{(\nu_j)_x(t,x;k)}$ is independent of t or x, then π_{ij}^* as well as Ψ_{ij} are also independent of t or x.

3.2 Calculating the Worst Crash Possible

In order to get the optimal portfolio strategy for an investor, who wants to maximize her expected final utility in the worst case possible, it is easier to calculate the portfolio strategy $\hat{\pi}$ first, which makes the investor crash indifferent. Obviously, the investor is indifferent towards a crash, if her maximized expected worst case final utility before a possible crash is equal to her maximized expected final utility after a crash of the worst possible case. That is, the investor's expected utility is not effected by a crash of the worst possible size. This justifies the following definitions, where the conventions

$$
\begin{aligned}
\hat{\nu}\,(t,x) &:= \mathbb{E}\left[U\left(X_0^{\hat{\pi},t,x}(T)\right)\right] \quad \text{and} \\
\bar{\nu}\,(t,x) &:= \mathbb{E}\left[U\left(X_0^{\bar{\pi},t,x}(T)\right)\right],
\end{aligned}
$$

which are the expected utility corresponding to the below described portfolio process $\hat{\pi}(t,x)$ and $\bar{\pi}(t,x)$, respectively, given the crash has not yet occurred at time t, are used.

Definition 3.6

i) A portfolio strategy $\hat{\pi}$ determined via the equation

$$
\hat{\nu}\,(t,x) = \inf_{k \in K \setminus \{0\}} \nu_1\,(t, x\,(1 - \hat{\pi}(t,x)k)\,; k) \quad \textit{for all } t \in [0,T] \textit{ and } x > 0
$$

will be called a **crash hedging strategy**.

ii) For $t \in [0,T]$ and $x > 0$ denote

$$k(t,x,\pi) := \left\{ \begin{array}{ll} 0 \quad \text{if } \nu_0(t,x) \le \nu_1(t, x(1-\pi(t,x)k); k) & \text{for all } k \in K \setminus \{0\} \\ \underset{k\in K\setminus\{0\}}{\arg\inf}\, \nu_1(t, x(1-\pi(t,x)k); k) & \text{else} \end{array} \right\}.$$

Thus $k(t,x,\pi)$ is the worst case size of a possible crash at time t for an investor with utility function U, wealth x at time t, and portfolio strategy π. Hence, k will be called the **worst crash size function** *and $k(t,x,\pi)$ will be the* **worst crash possible** *for π given U. In the following, often $\bar{k}(t,x)$ or $\bar{k}$ will be written instead of $k(t,x,\bar{\pi})$. Moreover, $k(t)$ or $k(t,x)$ will be used as an abbreviation for $k(t,x,\pi)$, if a statement holds for arbitrary $x > 0$ and/or $\pi \in A_0(t,x)$.*

iii) For $t \in [0,T]$ and $x > 0$ denote

$$\hat{k}(t,x) := \underset{k\in K\setminus\{0\}}{\arg\inf}\, \nu_1(t, x(1-\hat{\pi}(t,x)k); k).$$

Sometimes, $\hat{k}$ will be written instead of $\hat{k}(t,x)$.

Observe that the worst crash size function k has the following properties.

Lemma 3.7
The worst crash size function k satisfies for arbitrary $x > 0$ and $\pi \in A(T,x)$

$$k(T) = k(T,x,\pi) = \left\{ \begin{array}{ll} k^* = \max\{k | k \in K\} & \text{for } \pi > 0 \\ 0 & \text{for } \pi < 0 \end{array} \right\}.$$

If $k(t_1,x,\pi) = 0$ for some $t_1 \in [0,T]$ and for fixed $x > 0$ and $\pi \in A(T,x)$, then

$$k(s,x,\pi) = 0 \left\{ \begin{array}{ll} \text{for all } s \in [0,t_1] & \text{if } \pi > 0 \text{ is constant} \\ \text{for all } s \in [t_1,T] & \text{if } \pi < 0 \text{ is constant} \end{array} \right\}.$$

Proof: It is irrelevant what the perspective after a crash of size k at time T will look alike, since the investment will be terminated in T. Thus, the worst crash size in T is either the highest crash size possible (if the portfolio strategy satisfies $\pi > 0$) or zero (if $\pi < 0$).

$k(t_1,x,\pi) = 0$ means, that – with this long investment horizon and with the better perspectives after a crash – the threat of a crash is not really a threat. A crash at this time would be more profitable for the investor than to have no crash. Since the market situation after a crash just depends on the crash height k, but not on the crash time τ, it is straightforward that $k(s,x,\pi) = 0$ for $s \le t_1$, if $k(t_1,x,\pi) = 0$ for fixed $x > 0$ and if $\pi \in A(T,x)$ is constant. □

If $\pi > 0$, it is not possible that k is identical to zero. As t converges to T, $k(t)$ will eventually be greater than zero. $k(t) > 0$ means that a crash is an imminent threat to the investor.

Unfortunately, $k(t)$ can not be calculated explicitly. We can only give a differential equation, which determines k.

Proposition 3.8
Let the initial wealth x and the portfolio strategy $\pi(t,x)$ be given. If ν_1 is differentiable in k with $(\nu_1)_k(t,x;k) \le 0$ for all $k \in K$, then $k(t,x) = k^*$. If $\nu_1(t,x;k_*) \le \nu_1(t,x;k)$ for all $k \in (k_*,k^*]$, then set $k(t,x) = k_*$. Likewise, if $\nu_1(t,x;k^*) \le \nu_1(t,x;k)$ for all $k \in [k_*,k^*)$, then set $k(t,x) = k^*$. Otherwise define $h(t,x)$ via

$$(\nu_1)_k(t,x(1-\pi h);h) = \pi x(\nu_1)_x(t,x(1-\pi h);h), \tag{72}$$

given that ν_1 is continuously differentiable in x and k. Moreover, if $(\nu_1)(t,x;k)$ is twice continuously differentiable in x and k, then h is minimal, if

$$\begin{aligned}(\nu_1)_{kk}(t,x(1-\pi h);h) - 2\pi x(\nu_1)_{xk}(t,x(1-\pi h);h)&\\ \ge\ -\pi^2x^2(\nu_1)_{xx}(t,x(1-\pi h);h).&\end{aligned} \tag{73}$$

If ν_1 is a $C^{1,2,2}$–function, then the derivative of h with respect to t is given by

$$h_t = \frac{\pi x(\nu_1)_{xt} - (\nu_1)_{kt} + x\pi_t\left(h(\nu_1)_{kx} + (\nu_1)_x - x\pi h(\nu_1)_{xx}\right)}{(\nu_1)_{kk} + \pi^2x^2(\nu_1)_{xx} - 2\pi x(\nu_1)_{xk}},$$

and the derivative of h with respect to x is given by

$$\begin{aligned}h_x &= \frac{(1-\pi h)\left[\pi x(\nu_1)_{xx} - (\nu_1)_{kx}\right] + \pi(\nu_1)_x}{(\nu_1)_{kk} + \pi^2x^2(\nu_1)_{xx} - 2\pi x(\nu_1)_{xk}}\\ &\quad + x\pi_x\frac{h(\nu_1)_{kx} + (\nu_1)_x - x\pi h(\nu_1)_{xx}}{(\nu_1)_{kk} + \pi^2x^2(\nu_1)_{xx} - 2\pi x(\nu_1)_{xk}},\end{aligned}$$

where the derivatives of ν_1 are evaluated at $(t,x(1-\pi h);h)$ and π, h and their derivatives are evaluated at (t,x). Finally, the optimal k can be derived from h according to the following formula

$$k(t,x) = \left\{\begin{array}{cl} k^* & \text{if } h(t,x) \ge k^* \\ h(t,x) & \text{if } k_* < h(t,x) < k^* \\ k_* & \text{if } 0 < h(t,x) \le k_* \\ 0 & \text{if } h(t,x) \le 0 \end{array}\right\},$$

given that h minimizes $(\nu_1)(t,x(1-\pi h);h)$.

Observe that the right side in formula (73) is greater or equal than zero, if assumption A.2 holds. Thus, the left side must be positive as well, if h is supposed to be minimal.

If ν_1 is not twice continuously differentiable in x and k, one has to verify that

$$\begin{aligned}\frac{\partial}{\partial k}\nu_1(t,x(1-\pi(t)k);k) \le 0 \quad \text{for } k \le h \quad \text{and}\\ \frac{\partial}{\partial k}(\nu_1)(t,x(1-\pi(t)k);k) \ge 0 \quad \text{for } k \ge h.\end{aligned}$$

This shows then that h is indeed minimal.
Proof: If $(\nu_1)(t,x;k)$ is decreasing for all $k \in K$, then k^* is obviously the worst crash possible for the investor. This shows the first assertion.

The necessary condition for a minimum of $(\nu_1)(t,x;k)$ over all $k \in \overset{\circ}{K} := (k_*, k^*)$ is given by

$$\frac{\partial}{\partial k}\nu_1(t, x(1-\pi h); h) = 0.$$

The left side is equal to

$$(\nu_1)_k(t, x(1-\pi h); h) - \pi x (\nu_1)_x (t, x(1-\pi h); h) =: g(t,x;\pi,h),$$

which shows the second statement. Keeping in mind that this is only the necessary condition, it remains to check the sufficient condition. Given that ν_1 is twice continuously differentiable in x and k, the sufficient condition for a minimum is

$$\begin{aligned} & & \frac{\partial^2}{\partial k^2}\nu_1(t, x(1-\pi h); h) &\geq 0 \\ &\iff & \frac{\partial}{\partial k}\left[(\nu_1)_k(t, x(1-\pi h); h) - \pi x (\nu_1)_x (t, x(1-\pi h); h)\right] &\geq 0 \\ &\iff & (\nu_1)_{kk}(t, x(1-\pi h); h) - 2\pi x (\nu_1)_{xk}(t, x(1-\pi h); h) & \\ & & &\geq -\pi^2 x^2 (\nu_1)_{xx}(t, x(1-\pi h); h). \end{aligned}$$

which gives the asserted inequality. Since $g(t,x;\pi,h) = 0$ for all $(t,x) \in [0,T] \times (0,\infty)$, it is also valid that $g_t(t,x;\pi,h) = 0$ and $g_x(t,x;\pi,h) = 0$, which gives the next two claims, if ν_1 is a $C^{1,2,2}$–function.

$$\begin{aligned} & & 0 &= g_t(t,x;\pi,h) \\ &\iff & 0 &= (\nu_1)_{kt} - x(\nu_1)_{kx}[\pi_t h + \pi h_t] + (\nu_1)_{kk} h_t - \pi_t x (\nu_1)_x \\ & & &\quad - \pi x (\nu_1)_{xt} + \pi x^2 (\nu_1)_{xx}[\pi_t h + \pi h_t] - \pi x (\nu_1)_{xk} h_t \\ &\iff & h_t &= \frac{\pi x (\nu_1)_{xt} - (\nu_1)_{kt} + \pi_t \left(xh(\nu_1)_{kx} + x(\nu_1)_x - x^2 \pi h (\nu_1)_{xx}\right)}{(\nu_1)_{kk} + \pi^2 x^2 (\nu_1)_{xx} - 2\pi x (\nu_1)_{xk}}, \end{aligned}$$

where the derivatives of ν_1 are evaluated at $(t, x(1-\pi h); h)$ and π and h and their derivatives are evaluated at (t,x). Furthermore

$$\begin{aligned} & & 0 &= g_x(t,x;\pi,h) \\ &\iff & 0 &= (\nu_1)_{kx}[1 - \pi h - x\pi_x h - x\pi h_x] + (\nu_1)_{kk} h_x - \pi_x x (\nu_1)_x - \pi (\nu_1)_x \\ & & &\quad - \pi x (\nu_1)_{xx}[1 - \pi h - x\pi_x h - x\pi h_x] - \pi x (\nu_1)_{xk} h_x \\ &\iff & h_x &= \frac{(1-\pi h)\left[\pi x (\nu_1)_{xx} - (\nu_1)_{kx}\right] + \pi (\nu_1)_x}{(\nu_1)_{kk} + \pi^2 x^2 (\nu_1)_{xx} - 2\pi x (\nu_1)_{xk}} \\ & & &\quad + x\pi_x \frac{h(\nu_1)_{kx} + (\nu_1)_x - x\pi h (\nu_1)_{xx}}{(\nu_1)_{kk} + \pi^2 x^2 (\nu_1)_{xx} - 2\pi x (\nu_1)_{xk}}, \end{aligned}$$

where the derivatives of ν_1 are evaluated at $(t, x(1-\pi h); h)$ and π and h and their derivatives are evaluated at (t, x).

The formula for k is straightforward. □

Remark 3.9

1. Observe that it is possible that k depends on x. This only means that the *individual* worst possible crash size for an individual investor might depend on his wealth level.

   Further, this means that this model might include implicitly some kind of *individually perceived liquidity risk* for some specific utility functions. However, this *individually perceived liquidity risk* depends on the total wealth of the investor and not only on the wealth invested in the risky asset.

2. Notice that k is independent of x if $(\nu_1)_k \leq 0$ or if

$$\pi_x = \frac{(1-\pi k)\left[\pi x (\nu_1)_{xx} - (\nu_1)_{kx}\right] + \pi (\nu_1)_x}{xk(\nu_1)_{kx} + x(\nu_1)_x - x^2 \pi k (\nu_1)_{xx}},$$

   whenever $(\nu_1)_k (t, x(1-\pi k); k) > 0$ and assumption A.1 is valid.

3.3 The Crash Hedging Strategy

Let us suppose for this section that assumption A.1 holds.

A straightforward and heuristic approach of determining the crash hedging strategy is to differentiate the defining equation with respect to t. This leads to

$$\begin{aligned}
& & \hat{\nu}_t(t,x) &= (\nu_1)_t - x\left[\hat{\pi}_t \hat{k} + \hat{\pi} \hat{k}_t\right](\nu_1)_x + \hat{k}_t (\nu_1)_k \\
&\Longleftrightarrow & \hat{\pi}_t \hat{k} &= \frac{(\nu_1)_t}{x(\nu_1)_x} - \frac{\hat{\nu}_t(t,x)}{x(\nu_1)_x} + \hat{k}_t\left[\frac{(\nu_1)_k}{x(\nu_1)_x} - \hat{\pi}\right] \\
&\Longleftrightarrow & \hat{\pi}_t &= -\frac{1}{\hat{k}}\left[\Psi_1 - \frac{\hat{\nu}_t(t,x)}{x(\nu_1)_x}\right] + \frac{\hat{k}_t}{\hat{k}}\left[\frac{(\nu_1)_k}{x(\nu_1)_x} - \hat{\pi}\right] \\
& & &= -\frac{1}{\hat{k}}\left[\Psi_1 - \frac{\hat{\nu}_t(t,x)}{x(\nu_1)_x}\right],
\end{aligned}$$

where the functions $\hat{\pi}$ and $\hat{k}$ have to be evaluated at (t, x) and Ψ_1 and the various differentials of ν_1 have to be evaluated at $\left(t, x\left[1-\hat{\pi}(t,x)\hat{k}(t,x)\right]; \hat{k}(t,x)\right)$. The last equation is due to equation (72) which holds for the worst crash size $\hat{k}$ and is equivalent to

$$\hat{\pi}(t,x) = \frac{(\nu_1)_k\left(t, x\left[1-\hat{\pi}(t,x)\hat{k}(t,x)\right]; \hat{k}(t,x)\right)}{x(\nu_1)_x\left(t, x\left[1-\hat{\pi}(t,x)\hat{k}(t,x)\right]; \hat{k}(t,x)\right)}.$$

Although this approach has been used to calculate the crash hedging strategy in the case of logarithmic utility, it has the drawback that the formula still contains $\hat{\nu}_t(t,x)$. Neither is it easy to determine $\hat{\nu}(t,x)$ in general, nor is anything known about $\hat{\nu}(t,x)$ or $\hat{\nu}_t(t,x)$.

Therefore, the following approach will be made.

Theorem 3.10
Let $\nu_1(t,x;k)$ be a $C^{1,2,2}$–function and $\hat{k}(t,x)$ a $C^{1,2}$–function. Moreover, suppose that

$$\begin{aligned}
&(\nu_1)_x(t,x;k)\,\hat{\pi}(t,x)\,x\left[1-\hat{\pi}(t,x)\cdot\hat{k}(t,x)\right],\\
&(\nu_1)_x(t,x;k)\,x^2\hat{\pi}(t,x)\,\hat{k}(t,x)\,\hat{\pi}_x(t,x),\\
&(\nu_1)_x(t,x;k)\,x^2\hat{\pi}^2(t,x)\,\hat{k}_x(t,x)\text{ , and}\\
&(\nu_1)_k(t,x;k)\,x\hat{\pi}(t,x)\,\hat{k}_x(t,x)
\end{aligned}$$

are P–a.s. square integrable in the following sense

$$\int_0^T \left[(\nu_1)_k\left(t, X_0^{\hat{\pi}}(t)\left[1-\hat{\pi}\left(t,X_0^{\hat{\pi}}(t)\right)\cdot\hat{k}\left(t,X_0^{\hat{\pi}}(t)\right)\right];\hat{k}\left(t,X_0^{\hat{\pi}}(t)\right)\right)\right.$$
$$\left. X_0^{\hat{\pi}}(t)\hat{\pi}\left(t,X_0^{\hat{\pi}}(t)\right)\hat{k}_x\left(t,X_0^{\hat{\pi}}(t)\right)\right]^2 dt \quad < \quad \infty \quad P\text{–a.s.}.$$

Furthermore, let there exist a $C^{1,2}$–function $\hat{\pi}$ which is a solution of

$$\begin{aligned}
\hat{\pi}_t \;=\;& \left(\frac{1}{\hat{k}}-\hat{\pi}\right)\left\{\frac{\sigma_0^2}{2}\frac{x\,(\nu_1)_{xx}}{(\nu_1)_x}\left(\left[1-x\frac{\hat{k}\hat{\pi}_x+\hat{\pi}\hat{k}_x}{\left(1-\hat{\pi}\hat{k}\right)^2}\right]\hat{\pi}-\pi_{01}^*\right)^2+\Psi_{01}-\Psi_1\right\}\\
&+\frac{\sigma_0^2}{2}\frac{x\hat{\pi}^2}{\hat{k}\left(1-\hat{\pi}\hat{k}\right)}\left\{2\hat{k}_x\frac{\mu_0-r_0}{\sigma_0^2}-\frac{x\hat{\pi}_{xx}\hat{k}}{1-\hat{\pi}\hat{k}}+\frac{(\nu_1)_{kk}}{(\nu_1)_x}\hat{k}_x^2\right\}-\frac{x\hat{\pi}_x r_0}{1-\hat{\pi}\hat{k}},
\end{aligned}\qquad (74)$$

where $\hat{\pi}$ and $\hat{k}$ and its derivatives are evaluated at $\left(t,\frac{x}{1-\hat{\pi}\hat{k}}\right)$ and the different derivatives of ν_1 as well as π_{01}^*, Ψ_{01}, and Ψ_1 are evaluated at $\left(t,x;\hat{k}\right)$ with $(t,x,k)\in[0,T)\times(0,\infty)\times K\setminus\{0\}$. Moreover, the boundary condition is

$$\hat{\pi}(T,x)=0 \qquad \text{for all } x>0\,. \qquad (75)$$

Then $\hat{\pi}$ is the crash hedging strategy for an investor in market 0 given the utility function U_i of the investor in market i ($i\in\{0,1\}$) and the market coefficents *of market 1 after a possible crash. The corresponding value function before a possible crash satisfies*

$$\begin{aligned}
\hat{\nu}(t,x;k) &= \nu_1\left(t, x\left[1-\hat{\pi}(t,x)\cdot\hat{k}(t,x)\right]\right) \\
&= \mathbb{E}^{t,x}\left[\nu_1\left(s, X_0^{\hat{\pi}}(s)\left[1-\hat{\pi}\left(s, X_0^{\hat{\pi}}(s)\right)\cdot\hat{k}\left(s, X_0^{\hat{\pi}}(s)\right)\right]\right)\right]
\end{aligned}$$

for $0 \le t \le s \le T$.

Proof: Due to the fact, that after an immediate crash of the worst crash size $\hat{k}$ the portfolio process will change to the optimal portfolio process in the crash–free setting, $\pi_1^*\left(t,x;\hat{k}\right)$, one obtains

$$\hat{\nu}(t,x) = \nu_1\left(t, x\left[1-\hat{\pi}(t,x)\hat{k}(t,x)\right]; \hat{k}(t,x)\right).$$

As $\nu_1\left(t,x;\hat{k}\right)$ is a $C^{1,2,2}$-function and since $\hat{\pi}(t,x)$ as well as $\hat{k}(t,x)$ are supposed to be $C^{1,2}$–functions, Itô's formula is applicable. Making use of the convention

$$\begin{aligned}
z(t) &:= X_0^{\hat{\pi}}(t)\left[1-\hat{\pi}\left(t, X_0^{\hat{\pi}}(t)\right)\cdot\hat{k}\left(t, X_0^{\hat{\pi}}(t)\right)\right] \qquad \text{with} \\
dz(t) &= \left[1-\hat{\pi}\left(t, X_0^{\hat{\pi}}(t)\right)\cdot\hat{k}\left(t, X_0^{\hat{\pi}}(t)\right)\right] dX_0^{\hat{\pi}}(t) \\
&\quad -X_0^{\hat{\pi}}(t)\cdot\hat{\pi}\left(t, X_0^{\hat{\pi}}(t)\right) d\hat{k}\left(t, X_0^{\hat{\pi}}(t)\right) \\
&\quad -X_0^{\hat{\pi}}(t)\cdot\hat{k}\left(t, X_0^{\hat{\pi}}(t)\right) d\hat{\pi}\left(t, X_0^{\hat{\pi}}(t)\right) \qquad \text{and} \\
Z(t) &:= \left(t, X_0^{\hat{\pi}}(t)\left[1-\hat{\pi}\left(t, X_0^{\hat{\pi}}(t)\right)\cdot\hat{k}\left(t, X_0^{\hat{\pi}}(t)\right)\right]; \hat{k}\left(t, X_0^{\hat{\pi}}(t)\right)\right) \qquad \text{with} \\
dZ(t) &= \left(dt, dz(t); d\hat{k}\left(t, X_0^{\hat{\pi}}(t)\right)\right),
\end{aligned}$$

Itô's formula leads to

$$\begin{aligned}
\nu_1(Z(s)) &= \nu_1\left(t, x\left[1-\hat{\pi}(t,x)\hat{k}(t,x)\right]; \hat{k}(t,x)\right) + \int_t^s (\nu_1)_t (Z(u))\, du \\
&\quad + \int_t^s (\nu_1)_x (Z(u))\, dz(u) + \frac{1}{2}\int_t^s (\nu_1)_{xx}(Z(u))\, d[z,z]_u \\
&\quad + \int_t^s (\nu_1)_k (Z(u))\, d\hat{k}\left(u, X_0^{\hat{\pi}}(u)\right) + \frac{1}{2}\int_t^s (\nu_1)_{kk}(Z(u))\, d\left[\hat{k},\hat{k}\right]_u \\
&= \nu_1\left(t, x\left[1-\hat{\pi}(t,x)\hat{k}(t,x)\right]; \hat{k}(t,x)\right) + \int_t^s (\nu_1)_t (Z(u))\, du
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^s (\nu_1)_x (Z(u)) \left[r_0 + \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right)(\mu_0 - r_0)\right] z(u)\, du \\
&-\int_t^s (\nu_1)_x (Z(u))\, X_0^{\hat{\pi}}(u) \hat{k}\left(u, X_0^{\hat{\pi}}(u)\right) \Big\{ \hat{\pi}_t\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad + \hat{\pi}_x\left(u, X_0^{\hat{\pi}}(u)\right) X_0^{\hat{\pi}}(u) \left[r_0 + \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right)(\mu_0 - r_0)\right] \\
&\qquad + \frac{\sigma_0^2}{2} \hat{\pi}_{xx}\left(u, X_0^{\hat{\pi}}(u)\right) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{\pi}^2\left(u, X_0^{\hat{\pi}}(u)\right) \Big\}\, du \\
&-\int_t^s (\nu_1)_x (Z(u))\, X_0^{\hat{\pi}}(u) \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) \Big\{ \hat{k}_t\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad + \hat{k}_x\left(u, X_0^{\hat{\pi}}(u)\right) X_0^{\hat{\pi}}(u) \left[r_0 + \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right)(\mu_0 - r_0)\right] \\
&\qquad + \frac{\sigma_0^2}{2} \hat{k}_{xx}\left(u, X_0^{\hat{\pi}}(u)\right) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{\pi}^2\left(u, X_0^{\hat{\pi}}(u)\right) \Big\}\, du \\
&+\int_t^s (\nu_1)_k (Z(u)) \Big\{ \hat{k}_t\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad + \hat{k}_x\left(u, X_0^{\hat{\pi}}(u)\right) X_0^{\hat{\pi}}(u) \left[r_0 + \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right)(\mu_0 - r_0)\right] \\
&\qquad + \frac{\sigma_0^2}{2} \hat{k}_{xx}\left(u, X_0^{\hat{\pi}}(u)\right) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{\pi}^2\left(u, X_0^{\hat{\pi}}(u)\right) \Big\}\, du \\
&+\frac{\sigma_0^2}{2} \int_t^s (\nu_1)_{xx} (Z(u))\, \hat{\pi}^2\left(u, X_0^{\hat{\pi}}(u)\right) \Big\{ z(u)^2 \\
&\qquad + \left(X_0^{\hat{\pi}}(u)\right)^4 \hat{k}^2\left(u, X_0^{\hat{\pi}}(u)\right) \hat{\pi}_x^2\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad + \left(X_0^{\hat{\pi}}(u)\right)^4 \hat{\pi}^2\left(u, X_0^{\hat{\pi}}(u)\right) \hat{k}_x^2\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad - 2z(u) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{k}\left(u, X_0^{\hat{\pi}}(u)\right) \hat{\pi}_x\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad - 2z(u) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) \hat{k}_x\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad + 2 \left(X_0^{\hat{\pi}}(u)\right)^4 \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) \hat{k}\left(u, X_0^{\hat{\pi}}(u)\right) \hat{\pi}_x\left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad \cdot \hat{k}_x\left(u, X_0^{\hat{\pi}}(u)\right) \Big\}\, du \\
&+\frac{\sigma_0^2}{2} \int_t^s (\nu_1)_{kk} (Z(u))\, \hat{k}_x^2\left(u, X_0^{\hat{\pi}}(u)\right) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{\pi}^2\left(u, X_0^{\hat{\pi}}(u)\right) du \\
&+\int_t^s (\nu_1)_x (Z(u))\, \sigma_0 \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) z(u)\, dW(u)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_t^s (\nu_1)_x (Z(u)) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{k}\left(u, X_0^{\hat{\pi}}(u)\right) \sigma_0 \hat{\pi}_x \left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad \cdot \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) \, dW(u) \\
&-\int_t^s (\nu_1)_x (Z(u)) \left(X_0^{\hat{\pi}}(u)\right)^2 \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) \sigma_0 \hat{k}_x \left(u, X_0^{\hat{\pi}}(u)\right) \\
&\qquad \cdot \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) \, dW(u) \\
&+\int_t^s (\nu_1)_k (Z(u)) X_0^{\hat{\pi}}(u) \sigma_0 \hat{k}_x \left(u, X_0^{\hat{\pi}}(u)\right) \hat{\pi}\left(u, X_0^{\hat{\pi}}(u)\right) \, dW(u).
\end{aligned}
$$

It is important that the value function $\nu_1(Z(s))$ is a Martingale, since only this Martingale property ensures that a portfolio strategy makes an investor crash indifferent on the time interval $[0, T]$. Otherwise, a portfolio strategy might make an investor at most crash indifferent at some time points.

However, $\nu_1(Z(s))$ is only a martingale, if the following differential equation holds.

$$
\begin{aligned}
&(\nu_1)_t + x(\nu_1)_x \Bigg\{ \left(1 - \hat{\pi}\hat{k}\right)\left[r_0 + \hat{\pi}(\mu_0 - r_0)\right] - \hat{k}\Bigg(\hat{\pi}_t + x\hat{\pi}_x\left[r_0 + \hat{\pi}(\mu_0 - r_0)\right] \\
&\qquad + \frac{\sigma_0^2}{2}x^2\hat{\pi}_{xx}\hat{\pi}^2\Bigg) - \hat{\pi}\left(\hat{k}_t + x\hat{k}_x\left[r_0 + \hat{\pi}(\mu_0 - r_0)\right] + \frac{\sigma_0^2}{2}x^2\hat{k}_{xx}\hat{\pi}^2\right)\Bigg\} \\
&+ \frac{\sigma_0^2}{2}x^2(\nu_1)_{xx}\hat{\pi}^2\left[1 - \hat{\pi}\hat{k} - x\hat{k}\hat{\pi}_x - x\hat{\pi}\hat{k}_x\right]^2 \\
&+ (\nu_1)_k\left\{\hat{k}_t + x\hat{k}_x\left[r_0 + \hat{\pi}(\mu_0 - r_0)\right] + \frac{\sigma_0^2}{2}x^2\hat{k}_{xx}\hat{\pi}^2\right\} + \frac{\sigma_0^2}{2}x^2(\nu_1)_{kk}\hat{k}_x^2\hat{\pi}^2 = 0,
\end{aligned}
$$

where $\hat{\pi}$ and $\hat{k}$ and its differentials are evaluated at (t, x) and the different differentials of ν_1 are evaluated at $\left(t, x\left(1 - \hat{\pi}\hat{k}\right); \hat{k}\right)$. Solving this equation with respect to $\hat{k}\hat{\pi}_t$, rearranging terms, and using the definition of the generalized earning potential lead to

$$
\begin{aligned}
\hat{k}\hat{\pi}_t &= \frac{(\nu_1)_t}{x(\nu_1)_x} + \left(1 - \hat{\pi}\hat{k} - x\hat{k}\hat{\pi}_x - x\hat{\pi}\hat{k}_x\right)\left[r_0 + \hat{\pi}(\mu_0 - r_0)\right] - \hat{\pi}\hat{k}_t \\
&\quad - \frac{\sigma_0^2}{2}x^2\hat{\pi}^2\left(\hat{\pi}_{xx}\hat{k} + \hat{k}_{xx}\hat{\pi}\right) + \frac{\sigma_0^2}{2}\frac{x(\nu_1)_{xx}}{(\nu_1)_x}\hat{\pi}^2\left[1 - \hat{\pi}\hat{k} - x\hat{k}\hat{\pi}_x - x\hat{\pi}\hat{k}_x\right]^2 \\
&\quad + \frac{(\nu_1)_k}{x(\nu_1)_x}\left\{\hat{k}_t + x\hat{k}_x\left[r_0 + \hat{\pi}(\mu_0 - r_0)\right] + \frac{\sigma_0^2}{2}x^2\hat{k}_{xx}\hat{\pi}^2\right\} + \frac{\sigma_0^2}{2}\frac{x(\nu_1)_{kk}}{(\nu_1)_x}\hat{k}_x^2\hat{\pi}^2 \\
&= \left(1 - \hat{\pi}\hat{k} - x\hat{k}\hat{\pi}_x - x\hat{\pi}\hat{k}_x\right)r_0 + \frac{\sigma_0^2}{2}\frac{x(\nu_1)_{xx}}{(\nu_1)_x}\left\{\hat{\pi}^2\left[1 - \hat{\pi}\hat{k} - x\hat{k}\hat{\pi}_x - x\hat{\pi}\hat{k}_x\right]^2\right.
\end{aligned}
$$

$$
\begin{aligned}
& +2\left(1-\hat{\pi}\hat{k}-x\hat{k}\hat{\pi}_x-x\hat{\pi}\hat{k}_x\right)\hat{\pi}\frac{\mu_0-r_0}{\sigma_0^2}\frac{(\nu_1)_x}{x\,(\nu_1)_{xx}}\Bigg\}+\frac{(\nu_1)_t}{x\,(\nu_1)_x} \\
& +\left\{\hat{k}_t+x\hat{k}_x\left[r_0+\hat{\pi}\left(\mu_0-r_0\right)\right]+\frac{\sigma_0^2}{2}x^2\hat{k}_{xx}\hat{\pi}^2\right\}\left(\frac{(\nu_1)_k}{x\,(\nu_1)_x}-\hat{\pi}\right) \\
& +x\hat{k}_x\hat{\pi}\left[r_0+\hat{\pi}\left(\mu_0-r_0\right)\right]-\frac{\sigma_0^2}{2}x^2\hat{\pi}^2\hat{\pi}_{xx}\hat{k}+\frac{\sigma_0^2}{2}\frac{x\,(\nu_1)_{kk}}{(\nu_1)_x}\hat{k}_x^2\hat{\pi}^2 \\
= \; & \frac{\sigma_0^2}{2}\frac{x\,(\nu_1)_{xx}}{(\nu_1)_x}\left(\left[1-\hat{\pi}\hat{k}-x\hat{k}\hat{\pi}_x-x\hat{\pi}\hat{k}_x\right]\hat{\pi}-\left[1-\hat{\pi}\hat{k}\right]\pi_{01}^*\right)^2 \\
& +\left(1-\hat{\pi}\hat{k}\right)\left(\Psi_{01}-\Psi_1\right) \\
& +x\hat{k}_x\hat{\pi}^2\left(\mu_0-r_0\right)-\frac{\sigma_0^2}{2}x^2\hat{\pi}^2\hat{\pi}_{xx}\hat{k}+\frac{\sigma_0^2}{2}\frac{x\,(\nu_1)_{kk}}{(\nu_1)_x}\hat{k}_x^2\hat{\pi}^2-x\hat{k}\hat{\pi}_x r_0 \\
& +\left\{\hat{k}_t+x\hat{k}_x\left[r_0+\hat{\pi}\left(\mu_0-r_0\right)\right]+\frac{\sigma_0^2}{2}x^2\hat{k}_{xx}\hat{\pi}^2\right\}\left(\frac{(\nu_1)_k}{x\,(\nu_1)_x}-\hat{\pi}\right),
\end{aligned}
$$

where $\hat{\pi}$ and $\hat{k}$ and its differentials are evaluated at (t,x) and the different differentials of ν_1 as well as π_{01}^*, Ψ_{01}, and Ψ_1 are evaluated at $\left(t,x\left(1-\hat{\pi}\hat{k}\right);\hat{k}\right)$.

Observe that the last product is always zero, since either $\hat{k}$ is constant (that is the left side is zero) or $\hat{k}$ satisfies (72) (that is the right side is zero). Moreover, using the substitution $y := x\left(1-\hat{\pi}\hat{k}\right)$ yields

$$
\begin{aligned}
\hat{\pi}_t \;=\; & \left(\frac{1}{\hat{k}}-\hat{\pi}\right)\left\{\frac{\sigma_0^2}{2}\frac{y\,(\nu_1)_{xx}}{(\nu_1)_x}\left(\left[1-y\frac{\hat{k}\hat{\pi}_x+\hat{\pi}\hat{k}_x}{\left(1-\hat{\pi}\hat{k}\right)^2}\right]\hat{\pi}-\pi_{01}^*\right)^2+\Psi_{01}-\Psi_1\right\} \\
& +\frac{\sigma_0^2}{2}\frac{y\hat{\pi}^2}{\hat{k}\left(1-\hat{\pi}\hat{k}\right)}\left\{2\hat{k}_x\frac{\mu_0-r_0}{\sigma_0^2}-\frac{y\hat{\pi}_{xx}\hat{k}}{1-\hat{\pi}\hat{k}}+\frac{(\nu_1)_{kk}}{(\nu_1)_x}\hat{k}_x^2\right\}-\frac{y\hat{\pi}_x r_0}{1-\hat{\pi}\hat{k}},
\end{aligned}
$$

where $\hat{\pi}$ and $\hat{k}$ and its differentials are evaluated at $\left(t,\frac{y}{1-\hat{\pi}\hat{k}}\right)$ and the different differentials of ν_1 as well as π_{01}^*, Ψ_{01}, and Ψ_1 are evaluated at $\left(t,y;\hat{k}\right)$. This is the first assertion. Due to the supposed square integrability, it is straightforward to show the Martingale property. $\square$

A close look reveals that the partial differential equation (74) becomes an ordinary differential equation if $\hat{\pi}$ as well as $\hat{k}$ are independent of the wealth of the investor (that is of the second variable x).

Proposition 3.11
If $(\nu_1)_k \leq 0$ or if

$$\pi_x = \frac{(1-\pi k)\left[\pi x\,(\nu_1)_{xx} - (\nu_1)_{kx}\right] + \pi\,(\nu_1)_x}{xk\,(\nu_1)_{kx} + x\,(\nu_1)_x - x^2\pi k\,(\nu_1)_{xx}},$$

whenever $(\nu_1)_k\,(t, x\,(1-\pi k)\,; k) > 0$, then $\hat{k}$ is independent of x. In this case the differential equation (74) reduces to

$$\begin{aligned}\hat{\pi}_t &= \left(\frac{1}{\hat{k}} - \hat{\pi}\right)\left\{\frac{\sigma_0^2}{2}\frac{x\,(\nu_1)_{xx}}{(\nu_1)_x}\left(\left[1 - \frac{x\hat{k}\hat{\pi}_x}{\left(1-\hat{\pi}\hat{k}\right)^2}\right]\hat{\pi} - \pi_{01}^*\right)^2 + \Psi_{01} - \Psi_1\right\} \\ &\quad - \frac{\sigma_0^2}{2}\frac{x^2\hat{\pi}^2\hat{\pi}_{xx}}{\left(1-\hat{\pi}\hat{k}\right)^2} - \frac{x\hat{\pi}_x r_0}{1-\hat{\pi}\hat{k}}. \end{aligned} \tag{76}$$

Furthermore, if additionally $\frac{x(\nu_1)_{xx}}{(\nu_1)_x}$ is independent of x, then the crash hedging strategy $\hat{\pi}$ is independent of x as well. More specific, the differential equation (74) reduces in this case to

$$\hat{\pi}_t = \left(\frac{1}{\hat{k}} - \hat{\pi}\right)\left[\frac{\sigma_0^2}{2}\frac{x\,(\nu_1)_{xx}}{(\nu_1)_x}\left(\hat{\pi} - \pi_{01}^*\right)^2 + \Psi_{01} - \Psi_1\right]. \tag{77}$$

The differential equation (77) has a unique solution, if $\frac{1}{\hat{k}}$ and $\frac{x(\nu_1)_{xx}}{(\nu_1)_x}$ are continuous and bounded.

Proof: The first part is a straightforward conclusion of Remark 3.9. For the second part, it suffices to verify that the right side of equation (77) is indeed independent of x. Since $\hat{k}$ is assumed to be independent of x, it remains to verify that π_{01}^*, Ψ_{01}, and Ψ_1 are also independent of x. However, this follows directly from their definitions, since it has been supposed that $\frac{x(\nu_1)_{xx}}{(\nu_1)_x}$ is independent of x.

A solution of the differential equation (77) exists and is unique given that $\frac{1}{\hat{k}}$ and $\frac{x(\nu_1)_{xx}}{(\nu_1)_x}$ are continuous and bounded. This can be shown as in the proof of Theorem 2.5, p. 14, since both $\hat{k}$ and $\frac{x(\nu_1)_{xx}}{(\nu_1)_x}$ are independent of x. □

Remark 3.12
Equation (74) is a system of partial differential equation of order 2 in $\hat{\pi}$ and $\hat{k}$. Equation (76) is a partial differential equation of order 2 for $\hat{\pi}$. Equation (77) is an ordinary differential equation of order 1 for $\hat{\pi}$.

Observe that the first parts of Theorem 2.5 hold also with the differential equation (77), if $\hat{k}$ as well as $\frac{x(\nu_1)_{xx}}{(\nu_1)_x}$ are constant. Only the optimality does not hold, since this has been proved by using the properties of the logarithmic utility extensively. For the boundary conditions, just replace σ_0^2 by $\sigma_0^2 \frac{x(\nu_1)_{xx}}{(\nu_1)_x}$. Moreover, it is straightforward to verify that the following corollary holds.

Corollary 3.13
If $\hat{k}$ and $\frac{x(\nu_1)_{xx}}{(\nu_1)_x}$ in equation (77) are constant, then it is possible to give the solution of differential equation (77). Replacing k^* by $\hat{k}$ and σ_0^2 by $\sigma_0^2 \frac{x(\nu_1)_{xx}}{(\nu_1)_x}$, the solution is given by either Proposition 2.11 (if $\Psi_1 \geq r_0$) or Corollary 2.12 (if $\Psi_1 < r_0$).

3.4 The Optimal Crash Hedging Strategy

This section is based on Korn and Menkens [10]. Let us suppose for this section that assumptions A.1 and A.2 hold. Moreover, let us consider only the situation of equation (77), thus assuming that $\hat{\pi}$ is deterministic. Furthermore, suppose that π_0^* is positive and that $\hat{k}$ is constant, that means that $\hat{k}(t) \equiv k^*$.

With these assumption the following can be shown.

Proposition 3.14
Suppose that there exists a continuously differentiable solution $\hat{\pi}$ of equation (77) with boundary condition (75). Assume further that

$$f(x, y; t) := (\nu_1)_x (t, x) [y - \hat{\pi}(t)] [\mu - r] x + \frac{1}{2} (\nu_1)_{xx} (t, x) \sigma^2 \left[y^2 - \hat{\pi}(t)^2\right] x^2 \quad (78)$$

is a concave function in (x, y) for all $t \in [0, T)$. Moreover, let the following implication be valid

$$\left.\begin{array}{c} \mathbb{E}^{0,x} \left[\hat{\nu}\left(t, X_0^{\pi}(t)\right)\right] \leq \mathbb{E}^{0,x} \left[\hat{\nu}\left(t, X_0^{\hat{\pi}}(t)\right)\right] \quad \textit{and} \quad \mathbb{E}^{0,x} [\pi(t)] \geq \hat{\pi}(t) \\ \textit{for some } t \in [0, T),\ \pi(\cdot) \in A(x). \\ \Longrightarrow \mathbb{E}^{0,x} \left[\nu_1\left(t, X_0^{\pi}(t) \left[1 - \pi(t) k^*\right]\right)\right] \leq \mathbb{E}^{0,x} \left[\hat{\nu}\left(t, X_0^{\hat{\pi}}(t)\right)\right] . \end{array}\right\} \quad (79)$$

Then, $\hat{\pi}(t)$ is indeed the optimal portfolio process before the crash in the portfolio problem with at most one crash. The optimal portfolio process after the crash has happened coincides with the optimal one in the crash free setting of market 1.

Proof:

i) To prove optimality of $\hat{\pi}(t)$ and that $\hat{\nu}(t, x)$ coincides with the value function, consider $\hat{\nu}\left(t, X_0^{\hat{\pi}}(t)\right)$ for an arbitrary admissible portfolio process $\pi(t)$. With the help of Itô's formula one arrives at

$$\begin{aligned} \hat{\nu}\left(t, X_0^{\pi}(t)\right) &= \nu_1\left(t, X_0^{\pi}(t) \left[1 - \hat{\pi}(t) k^*\right]\right) \\ &= \nu_1\left(0, x \left[1 - \hat{\pi}(0) k^*\right]\right) + \int_0^t (\nu_1)_t \left(u, Z\left(u, \pi, \hat{\pi}\right)\right) du \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t (\nu_1)_x \left(u, Z\left(u,\pi,\hat{\pi}\right)\right) \Big[\left[r + \pi(u)\left(\mu - r\right)\right] Z\left(u,\pi,\hat{\pi}\right) \\
&\qquad -X_0^\pi(u)\hat{\pi}'(u)k^* \Big]\, du \\
&+\int_0^t \frac{1}{2} (\nu_1)_{xx} \left(u, Z\left(u,\pi,\hat{\pi}\right)\right) \sigma^2 \pi(u)^2 Z\left(u,\pi,\hat{\pi}\right)^2 \, du \\
&+\int_0^t (\nu_1)_x \left(u, Z\left(u,\pi,\hat{\pi}\right)\right) \sigma\pi(u) Z\left(u,\pi,\hat{\pi}\right) \, dW(u)\,,
\end{aligned}
$$

where the abbreviation

$$Z\left(t,\pi,\hat{\pi}\right) := X_0^\pi(t)\left[1 - \hat{\pi}(t)k^*\right]$$

has been used. If the differential equation (77) characterizing $\hat{\pi}(t)$ for the pairs

$$(t,x) = \left(u, X_0^\pi(u)\left[1 - \hat{\pi}(u)k^*\right]\right) = \left(u, Z\left(u,\pi,\hat{\pi}\right)\right)$$

in (77) is applied to replace $-X_0^\pi(u)\hat{\pi}'(u)k^*$ in the equation above and simplifying it afterwards, one obtains

$$
\begin{aligned}
\hat{\nu}\left(t, X_0^\pi(t)\right) &= \nu_1\left(0, x\left[1 - \hat{\pi}(0)k^*\right]\right) \\
&+ \int_0^t (\nu_1)_x \left(u, Z\left(u,\pi,\hat{\pi}\right)\right) \left[\pi(u) - \hat{\pi}(u)\right]\left[\mu - r\right] Z\left(u,\pi,\hat{\pi}\right) \, du \\
&+ \int_0^t \frac{1}{2} (\nu_1)_{xx} \left(u, Z\left(u,\pi,\hat{\pi}\right)\right) \sigma^2 \left[\pi(u)^2 - \hat{\pi}(u)^2\right] Z\left(u,\pi,\hat{\pi}\right)^2 \, du \\
&+ \int_0^t (\nu_1)_x \left(u, Z\left(u,\pi,\hat{\pi}\right)\right) \sigma\pi(u) Z\left(u,\pi,\hat{\pi}\right) \, dW(u)\,.
\end{aligned}
$$

In order to prove optimality of the portfolio process $\hat{\pi}(t)$, in the following only portfolio processes $\pi(t)$ will be considered that might yield a higher worst–case bound than $\hat{\pi}(t)$. A necessary condition for $\pi(t)$ to yield a higher worst–case bound is of course

$$\pi(0) < \hat{\pi}(0)$$

as otherwise the worst–case bound corresponding to $\pi(t)$ can at most equal the one corresponding to $\hat{\pi}(t)$.

ii) Assume now that there exists an admissible portfolio process $\pi(t)$ yielding a higher worst–case bound than $\hat{\pi}(t)$. As the inequality

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(s, X_0^{\pi}(s)\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(s, X_0^{\hat{\pi}}(s)\right)\right] \quad \text{for all } 0 \leq s \leq T, x > 0$$

would imply the non–existence of a higher worst–case bound for $\pi(t)$ (due to $\hat{\nu}(T,x) = U(x)$), it can be assumed that

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(s, X_0^{\pi}(s)\right)\right] > \mathbb{E}^{0,x}\left[\hat{\nu}\left(s, X_0^{\hat{\pi}}(s)\right)\right] \quad \text{for at least some } s > 0\,, \qquad (80)$$

and that this holds in particular for $s = T$. This then leads to

$$\begin{aligned}\mathbb{E}\Big[&(\nu_1)_x\left(s, Z\left(s,\pi,\hat{\pi}\right)\right)\left[\pi(s) - \hat{\pi}(s)\right]\left[\mu - r\right] Z\left(s,\pi,\hat{\pi}\right) \\ &+ \tfrac{1}{2}\left(\nu_1\right)_{xx}\left(s, Z\left(s,\pi,\hat{\pi}\right)\right)\sigma^2\left[\pi(s)^2 - \hat{\pi}(s)^2\right] Z\left(s,\pi,\hat{\pi}\right)^2\Big] > 0\end{aligned} \qquad (81)$$

for some $s > 0$. By assumption (78) and Jensen's inequality of the form $\mathbb{E}\left[f\left(X,Y\right)\right] \leq f\left(\mathbb{E}\left[X\right], \mathbb{E}\left[Y\right]\right)$ for concave functions applied to (81) with the choice

$$X := Z\left(s,\pi,\hat{\pi}\right) = X_0^{\pi}(s)\left[1 - \hat{\pi}(s)k^*\right] \quad \text{and} \quad Y := \pi(s)\,,$$

one gets

$$\begin{aligned}0 < &(\nu_1)_x\left(s, \mathbb{E}\left[Z\left(s,\pi,\hat{\pi}\right)\right]\right)\left\{\mathbb{E}\left[\pi(s)\right] - \hat{\pi}(s)\right\}\left[\mu - r\right]\mathbb{E}\left[Z\left(s,\pi,\hat{\pi}\right)\right] \\ &+ \tfrac{1}{2}\left(\nu_1\right)_{xx}\left(s, \mathbb{E}\left[Z\left(s,\pi,\hat{\pi}\right)\right]\right)\sigma^2\left[\mathbb{E}\left[\pi(s)\right]^2 - \hat{\pi}(s)^2\right]\mathbb{E}\left[Z\left(s,\pi,\hat{\pi}\right)\right]^2\end{aligned} \qquad (82)$$

for some $s > 0$. Due to the HJB–equation for the portfolio problem of the crash-free setting and to equations (77) and (75) one must have

$$\hat{\pi}(s) \leq \pi^*(s) \quad \text{for all } s \in [0,T]\,. \qquad (83)$$

This and the fact that $\pi^*(s)$ maximizes the right side of equation (82) (interpreted as a quadratic function in the variable $\mathbb{E}\left[\pi(s)\right]$) lead to either a contradiction in the case, when equality in (83) holds or to

$$\hat{\pi}(s) < \mathbb{E}\left[\pi(s)\right]\,. \qquad (84)$$

iii) For an arbitrary admissible portfolio process $\pi(t)$ yielding a higher worst–case bound than $\hat{\pi}(t)$ let

$$\bar{t} := \inf\left\{t > 0 \middle| \mathbb{E}\left[\pi(t)\right] \geq \hat{\pi}(t)\right\}\,. \qquad (85)$$

Case 1: Assume first that $0 < \bar{t} < T$. With this the following holds

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, X_0^{\pi}\left(\bar{t}\right)\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, X_0^{\hat{\pi}}\left(\bar{t}\right)\right)\right]\,,$$

which together with assumption (79) implies

$$\begin{aligned}\mathbb{E}^{0,x}\left[\nu_1\left(\bar{t}, X_0^{\pi}\left(\bar{t}\right)\left[1-\pi\left(\bar{t}\right)k^*\right]\right)\right] &\leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, X_0^{\hat{\pi}}\left(\bar{t}\right)\right)\right] \\ &= \mathbb{E}^{0,x}\left[\hat{\nu}\left(T, X_0^{\hat{\pi}}\left(T\right)\right)\right] \\ &= \mathbb{E}\left[U\left(X_0^{\hat{\pi}}\left(T\right)\right)\right] ,\end{aligned}$$

if the infimum defining $\bar{t}$ is indeed attained. If it is not attained, then the above inequality together with (79) implies

$$\begin{aligned}\mathbb{E}^{0,x}\left[\nu_1\left(\check{t}, X_0^{\pi}\left(\check{t}\right)\left[1-\pi\left(\check{t}\right)k^*\right]\right)\right] &\leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\check{t}, X_0^{\hat{\pi}}\left(\check{t}\right)\right)\right] \\ &= \mathbb{E}^{0,x}\left[\hat{\nu}\left(T, X_0^{\hat{\pi}}\left(T\right)\right)\right] \\ &= \mathbb{E}\left[U\left(X_0^{\hat{\pi}}\left(T\right)\right)\right]\end{aligned}$$

with $\check{t} = \bar{t} + \varepsilon$ for a suitable $\varepsilon > 0$ (see Lemma 3.15 below). Thus, both cases are contradicting the assumption that $\pi(t)$ yields a higher worst–case bound than $\hat{\pi}(t)$.

Case 2: In the case of $\bar{t} = T$ one would directly obtain

$$\begin{aligned}\mathbb{E}\left[U\left(X_0^{\pi}\left(T\right)\right)\right] &= \mathbb{E}^{0,x}\left[\nu_1\left(\bar{t}, X_0^{\pi}\left(\bar{t}\right)\left[1-\pi\left(\bar{t}\right)k^*\right]\right)\right] \\ &\leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, X_0^{\hat{\pi}}\left(\bar{t}\right)\right)\right] \\ &= \mathbb{E}\left[U\left(X_0^{\hat{\pi}}\left(T\right)\right)\right] ,\end{aligned}$$

again a contradiction to the assumption that $\pi(t)$ yields a higher worst–case bound than $\hat{\pi}(t)$.

Case 3: In the case of $\bar{t} = 0$ a contradiction to the assumption that $\pi(t)$ yields a higher worst–case bound than $\hat{\pi}(t)$ will be obtained. To see this note that the assumption of a higher worst–case bound for $\pi(t)$ can only be satisfied, if one has

$$\begin{aligned}\mathbb{E}^{0,x}\left[\nu_1\left(t, X_0^{\pi}(t)\left[1-\pi\left(t\right)k^*\right]\right)\right] &> \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, X_0^{\hat{\pi}}\left(t\right)\right)\right] \\ &= \mathbb{E}^{0,x}\left[\hat{\nu}\left(T, X_0^{\hat{\pi}}\left(T\right)\right)\right] \\ &= \mathbb{E}\left[U\left(X_0^{\hat{\pi}}\left(T\right)\right)\right] ,\end{aligned}$$

for all $0 < t \leq T$. On the other hand, for $t \downarrow 0$ the LCRL–property and the boundedness of $\hat{\pi}(t)$ and $\pi(t)$ together with the dominated convergence theorem imply

$$\begin{aligned}\nu_1\left(0, x\left[1-\hat{\pi}(0)k^*\right]\right) &= \lim_{t\downarrow 0}\mathbb{E}\left[\hat{\nu}\left(t, X_0^{\hat{\pi}}(t)\right)\right] \\ \mathbb{E}\left[\nu_1\left(0, x\left[1-\pi(0+)k^*\right]\right)\right] &= \lim_{t\downarrow 0}\mathbb{E}\left[\nu_1\left(t, X_0^{\pi}(t)\left[1-\pi(t)k^*\right]\right)\right] .\end{aligned}$$

The concavity of ν_1 together with the just stated limiting relations now lead to

$$\nu_1\left(0, x\left[1-\hat{\pi}(0)k^*\right]\right) \leq \mathbb{E}\left[\nu_1\left(t, x\left[1-\pi(0+)k^*\right]\right)\right]$$

$$\leq \quad \nu_1 \left(0, x \left[1 - \mathbb{E}\left[\pi(0+)\right] k^*\right]\right) .$$

But by the definition of $\bar{t}$ and the assumed strict concavity this can only be true, if

$$\pi(0+) = \hat{\pi}(0) \quad \text{a.s.}$$

which then contradicts the assumption that $\pi(\cdot)$ yields a higher worst–case bound than $\hat{\pi}(\cdot)$.

Putting all three cases together, it has been proved that there is no admissible portfolio process $\pi(t)$ yielding a higher worst–case bound than $\hat{\pi}(t)$.

□

Lemma 3.15
Let $\pi(t)$ be an admissible portfolio process satisfying condition (75) and let

$$\bar{t} := \inf \left\{ t > 0 \middle| \mathbb{E}\left[\pi(t)\right] \geq \hat{\pi}(t) \right\} .$$

If $0 < \bar{t} < T$ holds, then, under assumption (79), there exists a suitable $\varepsilon \geq 0$ such that for $\check{t} = \bar{t} + \varepsilon$ one has

$$\mathbb{E}^{0,x}\left[\nu_1\left(\check{t}, X_0^{\pi}\left(\check{t}\right)\left[1 - \pi\left(\check{t}\right) k^*\right]\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\check{t}, X_0^{\hat{\pi}}\left(\check{t}\right)\right)\right] .$$

Proof: In case of

$$\mathbb{E}\left[\pi\left(\bar{t}\right)\right] \geq \hat{\pi}\left(\bar{t}\right) , \tag{86}$$

the assertion is directly implied by assumption (79) for $\varepsilon = 0$. So let (86) be violated. As $0 < \bar{t} < T$ is valid there is a $\delta > 0$ with

$$\delta < \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, X_0^{\hat{\pi}}\left(\bar{t}\right)\right)\right] - \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t}, X_0^{\pi}\left(\bar{t}\right)\right)\right] ,$$

which can be concluded by part ii) in the proof of Proposition 3.14. But then continuity of $X_0^{\hat{\pi}}(t)$ and of $X_0^{\pi}(t)$ imply that there exists an $\varepsilon > 0$ such that the following holds

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t} + \varepsilon, X_0^{\pi}\left(\bar{t} + \varepsilon\right)\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(\bar{t} + \varepsilon, X_0^{\hat{\pi}}\left(\bar{t} + \varepsilon\right)\right)\right]$$

and the assertion then is a consequence of assumption (79). □

Corollary 3.16
Let $\hat{\pi}(\cdot)$ be the unique solution of (77). Moreover, assume that $\nu_1(t, x)$ is strictly increasing in x and strictly concave in x. Then it is the best possible deterministic portfolio.

Proof: If only deterministic portfolio strategies are considered, equation (81) reduces to

$$\begin{aligned} h(\pi) = & \ \mathbb{E}\left[(\nu_1)_x \left(s, Z\left(s, \pi, \hat{\pi}\right)\right) Z\left(s, \pi, \hat{\pi}\right)\right] \left[\pi(s) - \hat{\pi}(s)\right] \left[\mu - r\right] \\ & + \tfrac{1}{2}\mathbb{E}\left[(\nu_1)_{xx} \left(s, Z\left(s, \pi, \hat{\pi}\right)\right) Z\left(s, \pi, \hat{\pi}\right)^2\right] \sigma^2 \left[\pi(s)^2 - \hat{\pi}(s)^2\right]. \end{aligned}$$

Obviously, $h(\hat{\pi}) = 0$ and the function assumes its maximum in

$$\pi^*\left(s, Z\left(s, \pi, \hat{\pi}\right)\right) = -\frac{\mathbb{E}\left[(\nu_1)_x \left(s, Z\left(s, \pi, \hat{\pi}\right)\right) Z\left(s, \pi, \hat{\pi}\right)\right]}{\frac{1}{2}\mathbb{E}\left[(\nu_1)_{xx} \left(s, Z\left(s, \pi, \hat{\pi}\right)\right) Z\left(s, \pi, \hat{\pi}\right)^2\right]} \frac{\mu - r}{\sigma^2}.$$

Furthermore, the function h is strictly increasing for $\pi < \pi^*$, strictly decreasing for $\pi > \pi^*$, and concave for all π. Since $\nu_1(t,x)$ is strictly increasing and strictly concave in x, $\pi^*(t,x)$ is strictly positive. This guarantees that $\hat{\pi} \leq \pi^*$, because otherwise $\hat{\pi}$ cannot be a solution of (77) ($\hat{\pi}(T) = 0$ would yield a contradiction to $\hat{\pi} \geq \pi^* > 0$). Thus, $\hat{\pi} \leq \pi^*$ implies $h(\pi) < 0$ for all $\pi < \hat{\pi}$. Observe now that condition (79) is given straightforward. Given

$$\mathbb{E}^{0,x}\left[\hat{\nu}\left(t, X_0^{\pi}(t)\right)\right] \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, X_0^{\hat{\pi}}(t)\right)\right] \quad \text{and} \quad \pi(t) \geq \hat{\pi}(t)$$

for some $t \in [0, T)$, $\pi(\cdot) \in A(x)$. This implies

$$\begin{aligned} \mathbb{E}^{0,x}\left[\nu_1\left(t, X_0^{\pi}(t)\left[1 - \pi(t)k^*\right]\right)\right] & \leq \mathbb{E}^{0,x}\left[\nu_1\left(t, X_0^{\pi}(t)\left[1 - \hat{\pi}(t)k^*\right]\right)\right] \\ & = \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, X_0^{\pi}(t)\right)\right] \\ & \leq \mathbb{E}^{0,x}\left[\hat{\nu}\left(t, X_0^{\hat{\pi}}(t)\right)\right]. \end{aligned}$$

The assertion follows now just as in the proof of Proposition 3.14, part iii). □

4 Conclusion and Outlook

Considering changing market conditions leads to interesting phenomena (see Section 2.10). Just to mention one phenomenon, in the case of an initial short market (namely $\pi_0^* < 0$), it can happen that the optimal portfolio strategy is completely different from the corresponding crash hedging strategy. Scrutinizing these phenomena analytically, the definition of the *utility growth potential* (see Definition 2.3) is very useful (see Theorem 2.5).

Calculating the cost and the potential benefits of the crash hedging strategy reveals that it can be beneficial for the investor to follow the crash hedging strategy (see Lemma 2.21).

It has already been pointed out that the investment horizon is very important in crash modelling. This thesis did so by defining the *crash hedging strategy* to be the portfolio strategy which balances out – at *any* time where the investor is invested – the expected utility of the final wealth of the investor between the case of no crash occurring and the one of a crash occurring.

Next to the optimal portfolio under the threat of a crash, developed in Korn and Wilmott [11], this is the only approach known to the author which takes the investment horizon into account. By doing so this model does not need the *worst–case concept* developed by Korn and Wilmott [11]. However, examining the optimality of the crash hedging strategy, it shows that the crash hedging strategy is – at most – optimal in the sense of the worst–case concept, thereby revealing its close relationship with the worst–case model.

The investment horizon is even taken into consideration in the constant crash hedging strategy and the best worst–case constant portfolio strategy developed in Section 2.4. In contrast to that, the traditional crash model, introduced in Section 2.14 is independent of the investment horion. However, the investment strategies which can be observed in practice (e.g. the investment schemes of German pension funds) are taking their investment horizon into account. These strategies are similar to the portfolio strategies developed in this thesis and in Korn and Wilmott [11].

The disadvantage of the crash hedging approach and the worst–case concept is that they actually mix the classical expected mean maximization method in "normal times" with the expected worst–case bound maximization for the possible crash situation. The next goal is to work in both situations with the same maximization principle. Since it does not look promising to apply one of the above mentioned maximization methods to both situations, the concept of the *q–quantile crash hedging strategy* (see Section 2.13) is more appropriate.

The q–quantile crash hedging strategy applies the well–known Value at Risk concept to portfolio optimization. In spite of the fact that the q–quantile crash hedging strategy approach needs additional information, the Value at Risk concept is a natural candidate for evaluating portfolios under the threat of a crash. Although it has been introduced in this thesis in the setting of crash modelling, it remains to future research to apply the q–quantile crash hedging strategy not only to the possible crash situation but also to the situation of “normal times”.

This approach would have the advantage that the investor can calculate her Value at Risk in advance by following some kind of a q–quantile portfolio strategy. This would be interesting not only for the banking industry but in particular for the insurance industry who is used to calculate its risks in advance rather then “on time” (for a comparison of the different concepts used in the banking industry and the insurance industry see Bühlmann [3]).

A Appendix

A.1 Proof of Proposition 2.11

Proof of Proposition 2.11:

Case i) Applying the partial fraction expansion to the left integral of equation (25) yields

$$\int_{t_0}^{t_1} \left[\frac{A}{\frac{1}{k^*} - \hat{\pi}(t)} + \frac{B_1}{\pi_0^* - \hat{\pi}(t)} + \frac{B_2}{(\pi_0^* - \hat{\pi}(t))^2} \right] d\hat{\pi}(t). \tag{87}$$

One has to calculate the constants with the method of indetermined coefficients.

$$\begin{aligned}
A\left(\pi_0^* - \hat{\pi}(t)\right)^2 &= A\left[\hat{\pi}^2(t) - 2\pi_0^*\hat{\pi}(t) + (\pi_0^*)^2\right], \\
B_1\left(\frac{1}{k^*} - \hat{\pi}(t)\right)\left(\pi_0^* - \hat{\pi}(t)\right) &= B_1\left[\hat{\pi}^2(t) - \left(\pi_0^* + \frac{1}{k^*}\right)\hat{\pi}(t) + \frac{\pi_0^*}{k^*}\right], \\
B_2\left(\frac{1}{k^*} - \hat{\pi}(t)\right). &
\end{aligned}$$

One gets three different equations with three unknown constants.

1) $A + B_1 = 0 \quad \Longleftrightarrow \quad A = -B_1.$

2) $-2\pi_0^* A - \left(\pi_0^* + \frac{1}{k^*}\right) B_1 - B_2 = 0 \quad \Longleftrightarrow \quad B_2 = -\left(\pi_0^* - \frac{1}{k^*}\right) A.$

3) $\left(\pi_0^*\right)^2 A + \frac{\pi_0^*}{k^*} B_1 + \frac{1}{k^*} B_2 = 1$. This is equivalent to

$$\begin{aligned}
\left[\left(\pi_0^*\right)^2 - \frac{\pi_0^*}{k^*} - \frac{1}{k^*}\left(\pi_0^* - \frac{1}{k^*}\right)\right] A &= 1 \\
\Longleftrightarrow \qquad \left[\left(\pi_0^*\right)^2 - 2\frac{\pi_0^*}{k^*} + \frac{1}{(k^*)^2}\right] A &= 1 \\
\Longleftrightarrow \qquad A &= \frac{1}{\left(\pi_0^* - \frac{1}{k^*}\right)^2}.
\end{aligned}$$

This implies

$$\begin{aligned}
B_1 &= \frac{-1}{\left(\pi_0^* - \frac{1}{k^*}\right)^2} \\
\text{and} \quad B_2 &= \frac{1}{\pi_0^* - \frac{1}{k^*}}.
\end{aligned}$$

The next step is to calculate the different integrals of (87).

a) The first integral gives

$$\int_{t_0}^{t_1} \frac{A}{\frac{1}{k^*} - \hat{\pi}(t)}\, d\hat{\pi}(t) = -A \cdot \ln\left(\frac{1}{k^*} - \hat{\pi}(t)\right)\Bigg|_{t_0}^{t_1}.$$

Taking the integral from t to T and using (14) yields

$$\begin{aligned} -A \cdot \ln\left(\frac{1}{k^*} - \hat{\pi}(s)\right)\Bigg|_{t}^{T} &= -A \cdot \left[\ln\left(\frac{1}{k^*}\right) - \ln\left(\frac{1}{k^*} - \hat{\pi}(t)\right)\right] \\ &= A \cdot \ln\left(1 - \hat{\pi}(t)k^*\right) \\ &= \frac{\ln\left(1 - \hat{\pi}(t)k^*\right)}{\left(\pi_0^* - \frac{1}{k^*}\right)^2}. \end{aligned}$$

b) The second integral in (87) is similar to the first integral and calculates to

$$\int_{t_0}^{t_1} \frac{B_1}{\pi_0^* - \hat{\pi}(t)}\, d\hat{\pi}(t) = -B_1 \cdot \ln\left(\pi_0^* - \hat{\pi}(t)\right)\Bigg|_{t_0}^{t_1}.$$

Taking the integral from t to T and using (14) yields

$$\begin{aligned} -B_1 \cdot \ln\left(\pi_0^* - \hat{\pi}(s)\right)\Bigg|_{t}^{T} &= -B_1 \cdot \left[\ln\left(\pi_0^*\right) - \ln\left(\pi_0^* - \hat{\pi}(t)\right)\right] \\ &= B_1 \cdot \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^*}\right) \\ &= \frac{-\ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^*}\right)}{\left(\pi_0^* - \frac{1}{k^*}\right)^2}. \end{aligned}$$

c) The last integral in (87) is

$$\int_{t_0}^{t_1} \frac{B_2}{\left(\pi_0^* - \hat{\pi}(t)\right)^2}\, d\hat{\pi}(t) = \frac{B_2}{\pi_0^* - \hat{\pi}(t)}\Bigg|_{t_0}^{t_1}.$$

Taking the integral from t to T and using (14) yields

$$\begin{aligned} \frac{B_2}{\pi_0^* - \hat{\pi}(s)}\Bigg|_{t}^{T} &= B_2 \left[\frac{1}{\pi_0^*} - \frac{1}{\pi_0^* - \hat{\pi}(t)}\right] \\ &= B_2 \frac{\pi_0^* - \hat{\pi}(t) - \pi_0^*}{\pi_0^*\left(\pi_0^* - \hat{\pi}(t)\right)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{B_2}{\pi_0^*}\frac{-\hat{\pi}(t)}{\pi_0^* - \hat{\pi}(t)} \\
&= \frac{-1}{\left(\pi_0^*\right)^2 - \frac{\pi_0^*}{k^*}}\frac{\hat{\pi}(t)}{\hat{\pi}(t) - \pi_0^*}.
\end{aligned}
$$

Thus, equation (25) gives in this case if one integrates from t to T

$$
\begin{aligned}
&-\ln\left(1 - \hat{\pi}(t)k^*\right) + \ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^*}\right) + \left(\pi_0^* - \frac{1}{k^*}\right)\frac{1}{\pi_0^*}\frac{\hat{\pi}(t)}{\hat{\pi}(t) - \pi_0^*} \\
&\qquad = \frac{\sigma_0^2}{2}\left(\pi_0^* - \frac{1}{k^*}\right)^2 (T - t)\,.
\end{aligned}
$$

This is an implicit equation for $\hat{\pi}$, which can not be solved explicitly. However, further reduction delivers

$$
\begin{aligned}
&\ln\left(1 - \frac{\hat{\pi}(t)}{\pi_0^*}\right) - \ln\left(1 - \hat{\pi}(t)k^*\right) + \left(1 - \frac{1}{k^*\pi_0^*}\right)\frac{\hat{\pi}(t)}{\hat{\pi}(t) - \pi_0^*} \\
&\qquad = \frac{\sigma_0^2}{2}\left(\pi_0^* - \frac{1}{k^*}\right)^2 (T - t) \\
\Longleftrightarrow\quad &\ln\left(\frac{1 - \frac{\hat{\pi}(t)}{\pi_0^*}}{1 - \hat{\pi}(t)k^*}\right) + \left(1 - \frac{1}{k^*\pi_0^*}\right)\frac{\hat{\pi}(t)}{\hat{\pi}(t) - \pi_0^*} \\
&\qquad = \frac{\sigma_0^2}{2}\left(\pi_0^* - \frac{1}{k^*}\right)^2 (T - t)\,.
\end{aligned}
$$

Obviously, it is straightforward to give the inverse of $\hat{\pi}$. Denoting the inverse by $t\left(\hat{\pi}\right)$, it is easy to verify that

$$
\begin{aligned}
t\left(\hat{\pi}\right) &= T - \frac{2}{\sigma_0^2\left(\pi_0^* - \frac{1}{k^*}\right)^2}\left[1 - \frac{1}{k^*\pi_0^*} + \frac{\pi_0^* - \frac{1}{k^*}}{\hat{\pi}(t) - \pi_0^*} + \ln\left(\frac{1}{\pi_0^* \cdot k^*}\right)\right. \\
&\qquad \left. + \ln\left(\frac{\hat{\pi}(t) - \pi_0^*}{\hat{\pi}(t) - \frac{1}{k^*}}\right)\right].
\end{aligned}
$$

Taking the derivative of t with respect to $\hat{\pi}$ gives

$$
\begin{aligned}
t'\left(\hat{\pi}\right) &= -\frac{2}{\sigma_0^2\left(\pi_0^* - \frac{1}{k^*}\right)^2}\left[-\frac{\pi_0^* - \frac{1}{k^*}}{\left(\hat{\pi}(t) - \pi_0^*\right)^2} + \frac{\hat{\pi}(t) - \frac{1}{k^*}}{\hat{\pi}(t) - \pi_0^*}\cdot\right. \\
&\qquad \left. \cdot\frac{\hat{\pi}(t) - \frac{1}{k^*} - \left(\hat{\pi}(t) - \pi_0^*\right)}{\left(\hat{\pi}(t) - \frac{1}{k^*}\right)^2}\right] \\
&= \frac{2}{\sigma_0^2\left(\pi_0^* - \frac{1}{k^*}\right)^2}\left[\frac{\pi_0^* - \frac{1}{k^*}}{\left(\hat{\pi}(t) - \pi_0^*\right)^2} - \frac{\pi_0^* - \frac{1}{k^*}}{\left(\hat{\pi}(t) - \pi_0^*\right)\left(\hat{\pi}(t) - \frac{1}{k^*}\right)}\right]
\end{aligned}
$$

$$= \frac{2}{\sigma_0^2\left(\pi_0^* - \frac{1}{k^*}\right)} \left[\frac{\hat{\pi}(t) - \frac{1}{k^*} - \left(\hat{\pi}(t) - \pi_0^*\right)}{\left(\hat{\pi}(t) - \pi_0^*\right)^2 \left(\hat{\pi}(t) - \frac{1}{k^*}\right)} \right]$$
$$= \frac{1}{\frac{\sigma_0^2}{2}\left(\hat{\pi}(t) - \frac{1}{k^*}\right)\left(\hat{\pi}(t) - \pi_0^*\right)^2},$$

which is negative for $\hat{\pi} < \frac{1}{k^*}$ and $\hat{\pi} \neq \pi_0^*$. Thus, $t(\hat{\pi})$ is strictly decreasing which means that its inverse $\hat{\pi}(t)$ exists and is uniquely determined.

Case ii) Applying the partial fraction expansion to the left integral of equation (25) yields in this case

$$\int_{t_0}^{t_1} \left[\frac{A}{\frac{1}{k^*} - \hat{\pi}(t)} + \frac{B}{\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} - \hat{\pi}(t)} + \frac{C}{\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} - \hat{\pi}(t)} \right] d\hat{\pi}(t). \tag{88}$$

One has to calculate the constants with the method of indetermined coefficients.

$$A\left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} - \hat{\pi}(t)\right)\left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} - \hat{\pi}(t)\right)$$
$$= A\left[\hat{\pi}^2(t) - 2\pi_0^*\hat{\pi}(t) + \left(\pi_0^*\right)^2 - \frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)\right],$$
$$B\left(\frac{1}{k^*} - \hat{\pi}(t)\right)\left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} - \hat{\pi}(t)\right)$$
$$= B\left[\hat{\pi}^2(t) - \left(\pi_0^* + \frac{1}{k^*} - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}\right)\hat{\pi}(t) + \frac{\pi_0^*}{k^*} - \frac{1}{k^*}\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}\right],$$
$$C\left(\frac{1}{k^*} - \hat{\pi}(t)\right)\left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)} - \hat{\pi}(t)\right)$$
$$= C\left[\hat{\pi}^2(t) - \left(\pi_0^* + \frac{1}{k^*} + \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}\right)\hat{\pi}(t) + \frac{\pi_0^*}{k^*} + \frac{1}{k^*}\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0 - \Psi_1\right)}\right].$$

One gets three different equations with three unknown constants.

1) $A+B+C=0 \quad \Longleftrightarrow \quad C=-A-B.$

2)

$$
\begin{aligned}
-2\pi_0^* A - \left(\pi_0^* + \frac{1}{k^*} - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}\right) B & \\
- \left(\pi_0^* + \frac{1}{k^*} + \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}\right) C &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Longleftrightarrow \qquad B\left(2\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}\right) &= \left(\pi_0^* - \frac{1}{k^*} - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}\right) A \\
\Longleftrightarrow \qquad B &= \frac{\pi_0^* - \frac{1}{k^*} - \sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}}{2\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}} A \\
\Longleftrightarrow \qquad B &= \left(\frac{\pi_0^* - \frac{1}{k^*}}{2\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}} - \frac{1}{2}\right) A.
\end{aligned}
$$

3)

$$
\begin{aligned}
\left((\pi_0^*)^2 - \frac{2}{\sigma_0^2}\left(\Psi-\Psi_1\right)\right) A + \left(\frac{\pi_0^*}{k^*} - \frac{1}{k^*}\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}\right) B & \\
+ \left(\frac{\pi_0^*}{k^*} + \frac{1}{k^*}\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)}\right) C &= 1.
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
\left[(\pi_0^*)^2 - \frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right)\right] A + \left(\frac{\pi_0^*}{k^*} - \frac{\sqrt{2}}{\sigma_0 k^*}\sqrt{\Psi_0-\Psi_1}\right) B & \\
- \left(\frac{\pi_0^*}{k^*} + \frac{\sqrt{2}}{\sigma_0 k^*}\sqrt{\Psi_0-\Psi_1}\right)(A+B) &= 1 \\
\Longleftrightarrow \qquad \left[(\pi_0^*)^2 - \frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right) - \frac{\pi_0^*}{k^*} - \frac{\sqrt{2}}{\sigma_0 k^*}\sqrt{\Psi_0-\Psi_1}\right] A & \\
-2\frac{\sqrt{2}}{\sigma_0 k^*}\sqrt{\Psi_0-\Psi_1} B &= 1 \\
\Longleftrightarrow \qquad \Bigg[(\pi_0^*)^2 - \frac{2}{\sigma_0^2}\left(\Psi_0-\Psi_1\right) - \frac{\pi_0^*}{k^*} - \frac{\sqrt{2}}{\sigma_0 k^*}\sqrt{\Psi_0-\Psi_1} &
\end{aligned}
$$

$$
\begin{aligned}
&& \left. -\frac{\pi_0^*}{k^*} + \frac{1}{(k^*)^2} + \frac{\sqrt{2}}{\sigma_0 k^*}\sqrt{\Psi_0 - \Psi_1}\right] A &= 1 \\
\Longleftrightarrow && \left[(\pi_0^*)^2 + \frac{1}{(k^*)^2} - \frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1) - 2\frac{\pi_0^*}{k^*}\right] A &= 1 \\
\Longleftrightarrow && \left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right)\left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right) A &= 1 \\
\Longleftrightarrow \quad A &= \frac{1}{\left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right)\left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right)} \\
\Longleftrightarrow \quad A &= \frac{1}{\left(\pi_0^* - \frac{1}{k^*}\right)^2 - \frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
B &= \frac{1}{2\sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}\left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right)} \quad \text{and} \\
C &= \frac{-1}{2\sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)}\left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right)}.
\end{aligned}
$$

The next step is to calculate the different integrals of (88).

a) The first integral is already known and is

$$
\begin{aligned}
&\int_t^T \frac{A}{\frac{1}{k^*} - \hat{\pi}(s)}\, d\hat{\pi}(s) \\
&= -A \cdot \ln\left(\frac{1}{k^*} - \hat{\pi}(s)\right)\Bigg|_t^T \\
&= A \cdot \ln\left(1 - \hat{\pi}(t)k^*\right) \\
&= \frac{\ln\left(1 - \hat{\pi}(t)k^*\right)}{\left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right)\left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \frac{1}{k^*}\right)}.
\end{aligned}
$$

b) The second integral is similar to the first.

$$
\begin{aligned}
&\int_t^T \frac{B}{\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \hat{\pi}(s)}\, d\hat{\pi}(s) \\
&= -B \cdot \ln\left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2}(\Psi_0 - \Psi_1)} - \hat{\pi}(s)\right)\Bigg|_t^T
\end{aligned}
$$

$$
\begin{aligned}
&= -B \cdot \left[\ln \left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} \right) \right. \\
&\qquad \left. - \ln \left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} - \hat{\pi}(t) \right) \right] \\
&= B \cdot \ln \left(1 - \frac{\hat{\pi}(t)}{\pi_0^* + \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)}} \right) \\
&= \frac{\ln \left(1 - \frac{\hat{\pi}(t)}{\pi_0^* + \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)}} \right)}{2\sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} \left(\pi_0^* + \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} - \frac{1}{k^*} \right)}.
\end{aligned}
$$

c) The last integral in (88) gives

$$
\begin{aligned}
&\int_t^T \frac{C}{\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} - \hat{\pi}(s)} \, d\hat{\pi}(s) \\
&= -C \cdot \ln \left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} - \hat{\pi}(s) \right) \Bigg|_t^T \\
&= -C \cdot \left[\ln \left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} \right) \right. \\
&\qquad \left. - \ln \left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} - \hat{\pi}(t) \right) \right] \\
&= C \cdot \ln \left(1 - \frac{\hat{\pi}(t)}{\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)}} \right) \\
&= \frac{-\ln \left(1 - \frac{\hat{\pi}(t)}{\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)}} \right)}{2\sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} \left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} - \frac{1}{k^*} \right)}.
\end{aligned}
$$

Thus, equation (25) gives in this case if one integrates from t to T

$$
\begin{aligned}
&- \ln (1 - \hat{\pi}(t)k^*) \, 2\sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} \\
&- \ln \left(1 - \frac{\hat{\pi}(t)}{\pi_0^* + \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)}} \right) \left(\pi_0^* - \sqrt{\frac{2}{\sigma_0^2} (\Psi_0 - \Psi_1)} - \frac{1}{k^*} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\ln\left(1-\frac{\hat{\pi}(t)}{\pi_0^*-\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}}\right)\left(\pi_0^*+\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}-\frac{1}{k^*}\right)\\
=\;&\frac{\sigma_0^2}{2}(T-t)\left(\pi_0^*+\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}-\frac{1}{k^*}\right)\\
&\left(\pi_0^*-\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}-\frac{1}{k^*}\right)2\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}.
\end{aligned}
$$

This is an implicit equation for $\hat{\pi}$, which can not be solved explicitly. However, further reduction delivers

$$
\begin{aligned}
&2\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}\ln\left(1-\hat{\pi}(t)k^*\right)\\
&+\ln\left(1-\frac{\hat{\pi}(t)}{\pi_0^*+\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}}\right)\left(\pi_0^*-\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}-\frac{1}{k^*}\right)\\
&-\ln\left(1-\frac{\hat{\pi}(t)}{\pi_0^*-\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}}\right)\left(\pi_0^*+\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}-\frac{1}{k^*}\right)\\
=\;&-\sigma_0\sqrt{2(\Psi_0-\Psi_1)}(T-t)\left(\pi_0^*+\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}-\frac{1}{k^*}\right)\\
&\left(\pi_0^*-\sqrt{\frac{2}{\sigma_0^2}(\Psi_0-\Psi_1)}-\frac{1}{k^*}\right).
\end{aligned}
$$

Obviously, it is straightforward to give the inverse of $\hat{\pi}$. Again, defining the inverse by $t(\hat{\pi})$ and making use of the convention (26) and (27), it can be written as

$$
\begin{aligned}
t(\hat{\pi}) \;=\;& T+\frac{1}{\Theta_+\Theta_-\Delta\sigma_0^2}\left[2\Delta\left\{\ln(k^*)+\ln\left(\frac{1}{k^*}-\hat{\pi}\right)\right\}\right.\\
&+\Theta_-\left\{\ln(\pi_0^*+\Delta-\hat{\pi})-\ln(\pi_0^*+\Delta)\right\}\\
&\left.-\Theta_+\left\{\ln(\pi_0^*-\Delta-\hat{\pi})-\ln(\pi_0^*-\Delta)\right\}\right].
\end{aligned}
$$

With this the derivative of t with regard to $\hat{\pi}$ calculates to

$$
\begin{aligned}
t'(\hat{\pi}) \;=\;& \frac{1}{\Theta_+\Theta_-\Delta\sigma_0^2}\left[\frac{-2\Delta}{\frac{1}{k^*}-\hat{\pi}}-\frac{\Theta_-}{\pi_0^*+\Delta-\hat{\pi}}+\frac{\Theta_+}{\pi_0^*-\Delta-\hat{\pi}}\right]\\
=\;& \frac{-1}{\Theta_+\Theta_-\Delta\sigma_0^2}\left[\frac{2\Delta}{\frac{1}{k^*}-\hat{\pi}}+\frac{\Theta_-(\pi_0^*-\Delta-\hat{\pi})-\Theta_+(\pi_0^*+\Delta-\hat{\pi})}{(\hat{\pi}-\pi_0^*)^2-\Delta^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-1}{\Theta_+\Theta_-\Delta\sigma_0^2}\left[\frac{2\Delta}{\frac{1}{k^*}-\hat{\pi}}+\frac{2\Delta\left(\hat{\pi}-2\pi_0^*+\frac{1}{k^*}\right)}{\left(\hat{\pi}-\pi_0^*\right)^2-\Delta^2}\right] \\
&= \frac{-2}{\Theta_+\Theta_-\sigma_0^2}\left[\frac{\left(\hat{\pi}-\pi_0^*\right)^2-\Delta^2+\left(\frac{1}{k^*}-\hat{\pi}\right)\left(\hat{\pi}-2\pi_0^*+\frac{1}{k^*}\right)}{\left(\frac{1}{k^*}-\hat{\pi}\right)\left[\left(\hat{\pi}-\pi_0^*\right)^2-\Delta^2\right]}\right] \\
&= \frac{-2}{\Theta_+\Theta_-\sigma_0^2}\cdot\frac{\left(\pi_0^*-\frac{1}{k^*}\right)^2-\Delta^2}{\left(\frac{1}{k^*}-\hat{\pi}\right)\left[\left(\hat{\pi}-\pi_0^*\right)^2-\Delta^2\right]} \\
&= \frac{-2}{\sigma_0^2\left(\frac{1}{k^*}-\hat{\pi}\right)\left[\left(\hat{\pi}-\pi_0^*\right)^2-\Delta^2\right]} \\
&= \frac{1}{\left(\hat{\pi}-\frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}-\pi_0^*\right)^2+\Psi_1-\Psi_0\right]},
\end{aligned}
$$

which is negative for $\hat{\pi} < \frac{1}{k^*}$ and, if $\pi_0^* \geq 0$, for $\hat{\pi} < \pi_0^* - \Delta$. Thus, $t\left(\hat{\pi}\right)$ is strictly decreasing which means that its inverse $\hat{\pi}(t)$ exists and is uniquely determined.

Case iii) In this case the partial fraction expansion to the left integral of equation (25) yields

$$\int_{t_0}^{t_1}\left[\frac{A}{\frac{1}{k^*}-\hat{\pi}(t)}+\frac{D\hat{\pi}(t)+E}{\hat{\pi}^2(t)-2\pi_0^*\hat{\pi}(t)+\frac{2}{\sigma_0^2}\left(\Psi_1-r_0\right)}\right]d\hat{\pi}(t). \tag{89}$$

One has to calculate the constants with the method of indetermined coefficients.

$$
\begin{aligned}
& A\left(\hat{\pi}^2(t)-2\pi_0^*\hat{\pi}(t)+\frac{2}{\sigma_0^2}\left(\Psi_1-r_0\right)\right), \\
\left(D\hat{\pi}(t)+E\right)\left(\frac{1}{k^*}-\hat{\pi}(t)\right) &= D\left(\frac{\hat{\pi}(t)}{k^*}-\hat{\pi}^2(t)\right)+E\left(\frac{1}{k^*}-\hat{\pi}(t)\right).
\end{aligned}
$$

One gets three different equations with three unknown constants.

1) $A - D = 0 \quad \Longleftrightarrow \quad A = D.$

2) $-2\pi_0^*A + \frac{1}{k^*}D - E = 0 \quad \Longleftrightarrow \quad E = -\left(2\pi_0^* - \frac{1}{k^*}\right)A.$

3) $\frac{2}{\sigma_0^2}\left(\Psi_1 - r_0\right)A + \frac{1}{k^*}E = 1.$ This is equivalent to

$$
\begin{aligned}
\left(\frac{2}{\sigma_0^2}\left(\Psi_1-r_0\right)-2\frac{\pi_0^*}{k^*}+\frac{1}{\left(k^*\right)^2}\right)A &= 1 \\
\Longleftrightarrow \quad \left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)\right]A &= 1
\end{aligned}
$$

$$\iff \qquad A = \frac{1}{\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}.$$

This implies

$$\begin{aligned} D &= \frac{1}{\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)} \quad \text{and} \\ E &= \frac{-\left(2\pi_0^* - \frac{1}{k^*}\right)}{\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}. \end{aligned}$$

The next step is to calculate the different integrals of (89).

a) The first integral is already known and is

$$\begin{aligned} \int_t^T \frac{A}{\hat{\pi}(s) - \frac{1}{k^*}}\, d\hat{\pi}(s) &= -A \cdot \ln\left(\hat{\pi}(s) - \frac{1}{k^*}\right)\Bigg|_t^T \\ &= A \cdot \ln\left(1 - \hat{\pi}(t)k^*\right) \\ &= \frac{\ln\left(1 - \hat{\pi}(t)k^*\right)}{\left(\pi_0^* - \frac{1}{k^*}\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}. \end{aligned}$$

b) The second integral is given by

$$\begin{aligned} &\int_t^T \frac{D\hat{\pi}(s) + E}{\hat{\pi}^2(s) - 2\pi_0^*\hat{\pi}(s) + \frac{2}{\sigma_0^2}\left(\Psi_1 - r_0\right)}\, d\hat{\pi}(s) \\ &= \Bigg[\frac{D}{2}\ln\left(\hat{\pi}^2(s) - 2\pi_0^*\hat{\pi}(s) + \frac{2}{\sigma_0^2}\left(\Psi_1 - r_0\right)\right) \\ &\quad + \frac{E + D\pi_0^*}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1 - r_0\right) - \left(\pi_0^*\right)^2}} \arctan\left(\frac{\hat{\pi}(s) - \pi_0^*}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1 - r_0\right) - \left(\pi_0^*\right)^2}}\right)\Bigg]\Bigg|_t^T \\ &= \frac{D}{2}\ln\left(\left(\pi_0^*\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)\right) \\ &\quad + \frac{E + D\pi_0^*}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}} \arctan\left(\frac{-\pi_0^*}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}}\right) \\ &\quad - \frac{D}{2}\ln\left(\left(\hat{\pi}(t) - \pi_0^*\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)\right) \\ &\quad - \frac{E + D\pi_0^*}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}} \arctan\left(\frac{\hat{\pi}(t) - \pi_0^*}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1 - \Psi_0\right)}}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{D}{2}\ln\left(\frac{\left(\hat{\pi}(t)-\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}{\left(\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}\right) \\
&\quad -\frac{E+D\pi_0^*}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}}\arctan\left(\frac{\hat{\pi}(t)\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}}{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)-\hat{\pi}(t)\pi_0^*+\left(\pi_0^*\right)^2}\right) \\
&= \frac{-\ln\left(\frac{\left(\hat{\pi}(t)-\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}{\left(\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}\right)}{2\left(\pi_0^*-\frac{1}{k^*}\right)^2+\frac{4}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)} \\
&\quad +\frac{\left(\pi_0^*-\frac{1}{k^*}\right)\arctan\left(\frac{\hat{\pi}(t)\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}}{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)-\hat{\pi}(t)\pi_0^*+\left(\pi_0^*\right)^2}\right)}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)\right]},
\end{aligned}
$$

where

$$\frac{2}{\sigma_0^2}\left(\Psi_1-r_0\right)-\left(\pi_0^*\right)^2=\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)$$

and

$$\arctan(x)-\arctan(y)=\arctan\left(\frac{x-y}{1+x\cdot y}\right)$$

has been used.

Thus, integrating from t to T, equation (25) yields

$$
\begin{aligned}
&\frac{\ln\left(\frac{\left(\hat{\pi}(t)-\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}{\left(\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}\right)-2\ln\left(1-\hat{\pi}(t)k^*\right)}{2\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)\right]} \\
&\frac{-2\left(\pi_0^*-\frac{1}{k^*}\right)\arctan\left(\frac{\hat{\pi}(t)\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}}{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)-\hat{\pi}(t)\pi_0^*+\left(\pi_0^*\right)^2}\right)}{\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)\right]} \\
&= \frac{\sigma_0^2}{2}\left(T-t\right) \\
\iff\quad & -2\ln\left(1-\hat{\pi}(t)k^*\right)\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)} \\
&+\ln\left(\frac{\left(\hat{\pi}(t)-\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}{\left(\pi_0^*\right)^2+\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}\right)\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 2\left(\pi_0^* - \frac{1}{k^*}\right)\arctan\left(\frac{\hat{\pi}(t)\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}}{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right) - \hat{\pi}(t)\pi_0^* + \left(\pi_0^*\right)^2}\right) \\
&= \sigma_0^2\left(T-t\right)\sqrt{\frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)}\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2 + \frac{2}{\sigma_0^2}\left(\Psi_1-\Psi_0\right)\right].
\end{aligned}
$$

Obviously, it is straightforward to give the inverse of $\hat{\pi}$. Once more, naming the inverse function $t\left(\hat{\pi}\right)$ and making use of the convention (28), this case gives

$$
\begin{aligned}
t\left(\hat{\pi}\right) &= T + \frac{1}{\Delta_1\sigma_0^2\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\Delta_1^2\right]}\left[2\Delta_1\left\{\ln\left(k^*\right)+\ln\left(\frac{1}{k^*}-\hat{\pi}\right)\right\}\right. \\
&\quad - \Delta_1\left\{\ln\left(\left(\hat{\pi}-\pi_0^*\right)^2+\Delta_1^2\right) - \ln\left(\left(\pi_0^*\right)^2+\Delta_1^2\right)\right\} \\
&\quad \left.+ 2\left(\pi_0^*-\frac{1}{k^*}\right)\left\{\arctan\left(\frac{\hat{\pi}-\pi_0^*}{\Delta_1}\right) - \arctan\left(\frac{-\pi_0^*}{\Delta_1}\right)\right\}\right].
\end{aligned}
$$

With this, the derivative of t with respect to $\hat{\pi}$ calculates to

$$
\begin{aligned}
t'\left(\hat{\pi}\right) &= \frac{1}{\Delta_1\sigma_0^2\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\Delta_1^2\right]}\left[\frac{-2\Delta_1}{\frac{1}{k^*}-\hat{\pi}} - \frac{2\Delta_1\left(\hat{\pi}-\pi_0^*\right)}{\left(\hat{\pi}-\pi_0^*\right)^2+\Delta_1^2}\right. \\
&\quad \left.+ \frac{2\Delta_1\left(\pi_0^*-\frac{1}{k^*}\right)}{\left(\hat{\pi}-\pi_0^*\right)^2+\Delta_1^2}\right] \\
&= \frac{-2}{\sigma_0^2\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\Delta_1^2\right]} \\
&\quad \cdot\frac{\left(\hat{\pi}-\pi_0^*\right)^2+\Delta_1^2+\left(\hat{\pi}-2\pi_0^*+\frac{1}{k^*}\right)\left(\frac{1}{k^*}-\hat{\pi}\right)}{\left(\frac{1}{k^*}-\hat{\pi}\right)\left[\left(\hat{\pi}-\pi_0^*\right)^2+\Delta_1^2\right]} \\
&= \frac{-2}{\sigma_0^2\left[\left(\pi_0^*-\frac{1}{k^*}\right)^2+\Delta_1^2\right]}\cdot\frac{\left(\pi_0^*-\frac{1}{k^*}\right)^2+\Delta_1^2}{\left(\frac{1}{k^*}-\hat{\pi}\right)\left[\left(\hat{\pi}-\pi_0^*\right)^2+\Delta_1^2\right]} \\
&= \frac{-2}{\sigma_0^2\left(\frac{1}{k^*}-\hat{\pi}\right)\left[\left(\hat{\pi}-\pi_0^*\right)^2+\Delta_1^2\right]} \\
&= \frac{1}{\left(\hat{\pi}-\frac{1}{k^*}\right)\left[\frac{\sigma_0^2}{2}\left(\hat{\pi}-\pi_0^*\right)^2+\Psi_1-\Psi_0\right]}.
\end{aligned}
$$

It is straightforward to see that the derivative of t is negative for $\hat{\pi} < \frac{1}{k^*}$. Thus, $t\left(\hat{\pi}\right)$ is strictly decreasing which means that its inverse $\hat{\pi}(t)$ exists and is uniquely determined.

Case iv) In this special case the partial fraction expansion to the left integral of equation (25) yields

$$\int_{t_0}^{t_1} \frac{d\hat{\pi}(t)}{\left(\hat{\pi}(t) - \pi_0^*\right)^3}. \tag{90}$$

It is easy to calculate this integral

$$\begin{aligned} \int_t^T \frac{d\hat{\pi}(t)}{\left(\hat{\pi}(t) - \pi_0^*\right)^3} &= -\frac{1}{2} \frac{1}{\left(\hat{\pi}(t) - \pi_0^*\right)^2} \Bigg|_t^T \\ &= -\frac{1}{2} \frac{1}{\left(\pi_0^*\right)^2} + \frac{1}{2} \frac{1}{\left(\hat{\pi}(t) - \pi_0^*\right)^2}. \end{aligned}$$

Hence, integrating from t to T, equation (25) yields

$$\begin{aligned} & \frac{1}{\left(\hat{\pi}(t) - \pi_0^*\right)^2} &= \frac{1}{\left(\pi_0^*\right)^2} + \sigma_0^2 (T - t) \\ \Longleftrightarrow \quad & \left(\hat{\pi}(t) - \pi_0^*\right)^2 &= \frac{\left(\pi_0^*\right)^2}{\left(\pi_0^*\right)^2 \sigma_0^2 (T - t) + 1} \\ \Longleftrightarrow \quad & \hat{\pi}_{1/2}(t) &= \pi_0^* \pm \frac{\pi_0^*}{\sqrt{\left(\pi_0^*\right)^2 \sigma_0^2 (T - t) + 1}}. \end{aligned}$$

Of these two functions only the second one satisfy $\hat{\pi}(T) = 0$, thus

$$\hat{\pi}(t) = \pi_0^* - \frac{\pi_0^*}{\sqrt{\left(\pi_0^*\right)^2 \sigma_0^2 (T - t) + 1}}.$$

This is the only case where it is possible to calculate the crash hedging strategy $\hat{\pi}$ explicitly. Note that $\hat{\pi}(t) < \pi_0^*$ for all $t \in [0, T]$.

Case v) This case as well as case vi) are analog to case i).

□

A.2 Calculating the Value Function $\nu_{P,\pi}$

If the price process of the risky asset evolves according to equation (60), the corresponding wealth equation is given as

$$
\begin{aligned}
dX_0^{\pi,t,x}(s) &= X_0^{\pi,t,x}(s)\Bigg[\left(r_0 + \pi(s)\left(\mu_0 - r_0\right)\right)\, ds + \pi(s)\sigma_0\, dW(s) \\
&\qquad - \pi(s)\sum_{j=1}^{m} k_j\, dN_j(s)\Bigg]\,, \\
X_0^{\pi,t,x}(t) &= x\,.
\end{aligned}
$$

Applying the generalized Itô–formula (see e.g. Klebaner [8], pp. 200, formula (8.22)) to $\ln\left(X_0^{\pi,t,x}(T)\right)$ yields

$$
\begin{aligned}
\ln\left(X_0^{\pi,t,x}(T)\right) &= \ln(x) + \int_t^T \frac{dX_0^{\pi,t,x}(s)}{X_0^{\pi,t,x}(s)} - \frac{1}{2}\int_t^T \frac{1}{\left(X_0^{\pi,t,x}(s)\right)^2}\, d\left[X_0^{\pi,t,x}, X_0^{\pi,t,x}\right](s) \\
&\quad + \sum_{t\le s\le T}\Bigg[\ln\left(X_0^{\pi,t,x}(s)\right) - \ln\left(X_0^{\pi,t,x}(s-)\right) - \frac{\triangle X_0^{\pi,t,x}(s)}{X_0^{\pi,t,x}(s-)} \\
&\qquad + \frac{1}{2}\frac{\left(\triangle X_0^{\pi,t,x}(s)\right)^2}{\left(X_0^{\pi,t,x}(s-)\right)^2}\Bigg] \\
&= \ln(x) + \int_t^T \left(r_0 + \pi(s)\left(\mu_0 - r_0\right)\right)\, ds + \int_t^T \pi(s)\sigma_0\, dW(s) \\
&\quad - \sum_{j=1}^{m}\int_t^T \pi(s)k_j\, dN_j(s) - \frac{1}{2}\int_t^T \pi^2(s)\sigma_0^2\, ds \\
&\quad - \frac{1}{2}\sum_{j=1}^{m}\int_t^T \pi^2(s)k_j^2\, dN_j(s) \\
&\quad + \sum_{t\le s\le T}\Bigg[\ln\left(X_0^{\pi,t,x}(s)\right) - \ln\left(X_0^{\pi,t,x}(s-)\right) \\
&\qquad + \frac{X_0^{\pi,t,x}(s-)}{X_0^{\pi,t,x}(s-)}\sum_{j=1}^{m}\pi(s)k_j \triangle N_j(s) \\
&\qquad + \frac{1}{2}\frac{\left(X_0^{\pi,t,x}(s-)\right)^2}{\left(X_0^{\pi,t,x}(s-)\right)^2}\sum_{j=1}^{m}\pi^2(s)k_j^2 \triangle N_j(s)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \ln(x) + \int_t^T \Psi_0 + (\pi(s) - \pi_0^*)^2 \, ds + \int_t^T \pi(s)\sigma_0 \, dW(s) \\
&\quad + \sum_{t \le s \le T} \left[\ln\left(X_0^{\pi,t,x}(s)\right) - \ln\left(X_0^{\pi,t,x}(s-)\right) \right].
\end{aligned}
$$

Thus, the wealth equation has the solution

$$
\begin{aligned}
X_0^{\pi,t,x}(T) &= x \exp\left(\int_t^T \Psi_0 + (\pi(s) - \pi_0^*)^2 \, ds + \int_t^T \pi(s)\sigma_0 \, dW(s) \right) \\
&\quad \cdot \prod_{t \le s \le T} \frac{X_0^{\pi,t,x}(s)}{X_0^{\pi,t,x}(s-)}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t \le s \le T} \left[\ln\left(X_0^{\pi,t,x}(s)\right) - \ln\left(X_0^{\pi,t,x}(s-)\right) \right] \right] \\
&= \mathbb{E}\left[\int_t^T \sum_{j=1}^m \left[\ln\left(X_0^{\pi,t,x}(s-) - X_0^{\pi,t,x}(s)\pi(s)k_j\right) - \ln\left(X_0^{\pi,t,x}(s-)\right) \right] \, dN_j(s) \right] \\
&= \mathbb{E}\left[\int_t^T \sum_{j=1}^m \left[\ln\left(X_0^{\pi,t,x}(s-) - X_0^{\pi,t,x}(s)\pi(s)k_j\right) - \ln\left(X_0^{\pi,t,x}(s-)\right) \right] \lambda_j(s) \, ds \right] \\
&= \mathbb{E}\left[\int_t^T \sum_{j=1}^m \ln(1 - \pi(s)k_j) \, \lambda_j(s) \, ds \right],
\end{aligned}
$$

where the last equation is due to the fact that the set of points $s \in [0, T]$ which satisfy $X_0^{\pi,t,x}(s-) \neq X_0^{\pi,t,x}(s)$ has Lebesgue–measure zero. With this, it is straightforward to derive the value function for the portfolio strategy π.

$$
\begin{aligned}
\nu_{P,\pi}(t, x) &= \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right) \right] \\
&= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2} (\pi(s) - \pi_0^*)^2 \right.\right. \\
&\qquad \left.\left. + \sum_{j=1}^m \ln(1 - \pi(s)k_j) \, \lambda_j(s) \right] \, ds \right].
\end{aligned}
$$

Assuming that $m = 1$, $k_1 = k$ and $\lambda_1 = \lambda$ is deterministic, the value function for the portfolio strategy π reduces to

$$\nu_{P,\pi}(t,x) = \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2 + \ln\left(1 - \pi(s)k\right)\lambda(s)\right] ds\right].$$

A.3 Calculating the Optimal Portfolio Strategy π_P^*

The optimal portfolio strategy π_P^* can be calculated explicitly only in the case $m = 1$, which will be done in the following. For the case $m > 1$ see e.g. Aase [1]. In order to find the optimal portfolio strategy π_P^* one has to find the π which maximizes

$$f(\pi) := \Psi_0 - \frac{\sigma_0^2}{2}\left(\pi - \pi_0^*\right)^2 + \ln\left(1 - \pi k\right)\lambda.$$

Clearly, this function is twice continuously differentiable with respect to π. Doing so gives

$$\begin{aligned} \frac{\partial f}{\partial \pi}(\pi) &= -\sigma_0^2\left(\pi - \pi_0^*\right) - \frac{k\lambda}{1 - \pi k} \qquad \text{and} \\ \frac{\partial^2 f}{\partial \pi^2}(\pi) &= -\sigma_0^2 - \frac{k^2\lambda}{\left(1 - \pi k\right)^2}. \end{aligned}$$

For $\lambda \geq 0$ the second derivative is always negative. Setting the first derivative equal to zero and solving it with respect to π yields

$$\begin{aligned} && \left(\pi - \pi_0^*\right)\left(1 - \pi k\right) &= -\frac{k\lambda}{\sigma_0^2} \\ \Longleftrightarrow && \pi^2 - \pi\left(\pi_0^* + \frac{1}{k}\right) &= \frac{\lambda}{\sigma_0^2} - \frac{\pi_0^*}{k} \\ \Longleftrightarrow && \left(\pi - \frac{1}{2}\left(\pi_0^* + \frac{1}{k}\right)\right)^2 &= \frac{1}{4}\left(\pi_0^* + \frac{1}{k}\right)^2 - \frac{\pi_0^*}{k} + \frac{\lambda}{\sigma_0^2} \\ \Longleftrightarrow && \pi_{1/2} &= \frac{1}{2}\left(\pi_0^* + \frac{1}{k}\right) \pm \sqrt{\frac{1}{4}\left(\pi_0^* - \frac{1}{k}\right)^2 + \frac{\lambda}{\sigma_0^2}}. \end{aligned}$$

Hence, both points maximize f with $\pi_1 > \pi_0^* > \pi_2$ for either $\frac{1}{k} \neq 0$ or $\lambda > 0$.

Remember that $k > 0$ determines a possible crash height while $k < 0$ describes a possible positive price jump of the risky asset. Thus, it is not rational for an investor to invest more in the risky asset given the possibility of a crash than in the crash–free situation. Likewise, it is not rational for an investor to invest less in the risky asset given the possibility of a sudden upward jump than in the crash–free situation. Hence, $\pi_P^* := \pi_2$ is the only rational optimal portfolio strategy.

References

[1] Knut Kristian Aase. Optimum portfolio diversification in a general continuous-time model. *Stochastic Processes and their Applications*, 18:81 – 98, 1984.

[2] Richard Ernest Bellman. *Dynamic Programming*. Princeton University Press, 1957.

[3] Hans Bühlmann. Mathematische Paradigmen in der Finanzwirtschaft. *Elemente der Mathematik*, 53:159 – 176, 1998.

[4] Ernst Eberlein and Ulrich Keller. Hyperbolic distribution in finance. *Bernoulli*, 1:281 – 299, 1995.

[5] Paul Embrechts, Claudia Klüppelberg, and Thomas Mikosch. *Modelling Extremal Events*. Springer, Berlin, 1997.

[6] Otto Forster. *Analysis 2*. vieweg studium, Grundkurs Mathematik. Friedr. Vieweg & Sohn, Braunschweig, 5. edition, 1987. ISBN 3–528–37231–1.

[7] Harro Heuser. *Gewöhnliche Differentialgleichungen: Einführung in Lehre und Gebrauch*. B. G. Teubner, Stuttgart, 1989. ISBN 3–519–02227-3.

[8] Fima C. Klebaner. *Introduction to Stochastic Calculus with Applications*. Imperial College Press, London, 1998.

[9] Ralf Korn. *Optimal Portfolios*. World Scientific, Singapore, first edition, 1997. ISBN 981-023215-2.

[10] Ralf Korn and Olaf Menkens. Worst-case scenario portfolio optimization: A new stochastic control approach. Unpublished Working Paper, April 2002.

[11] Ralf Korn and Paul Wilmott. Optimal portfolios under the threat of a crash. *International Journal of Theoretical and Applied Finance*, 5(2):171 – 187, 2002.

[12] Robert C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125 – 144, January – March 1976.

[13] Robert C. Merton. *Continuous-time Finance*. Blackwell Publishers Ltd, Oxford, revised edition, 1990. ISBN 0-631-15847-2, or 0-631-18508-9.

Academic Career (in Short)

Academic background:	since Oct 2004	Lecturer in Computational Finance and Ecomomic Agents in the Center of Computational Finance and Ecomomic Agents (CCFEA) at the Department of Economics of the University of Essex, United Kingdom.
	July 2004	MSc ("Diplom") in Economics. MSc thesis "Value at Risk and Self–Similarity" written under the supervision of Prof. Klaus Schürger at the University of Bonn, Germany.
	May 2001 until Oct 2003	Research associate ("wissenschaftlicher Mitarbeiter") of Prof. Ralf Korn in the Department of Mathematics at the Technical University of Kaiserslautern, Germany within the Schwerpunkt Programm (SPP) 1033 "Interagierende stochastische Systeme von hoher Komplexität" of the Deutsche Forschungsgemeinschaft (DFG).
	Oct 1998 until Dec 2000	Research associate ("wissenschaftlicher Mitarbeiter") in the Department of Mathematics at the University of Bremen, Germany.
	Jan 1997	MSc ("Diplom") in Mathematics. MSc thesis "Insider Trading in Continuous Time" written under the supervision of Prof. Hans Föllmer at the University of Bonn, Germany.
	Oct 1991 until Dec 1996	Studying Mathematics und Economics at the University of Bonn, Germany.
	Sep 1990 until July 1991	Studied at the University of California, Davis, USA. Partial grant from the education abroad program (EAP) from the University of California.
	July 1990	BSc ("Vordiplom") in Economics.
	April 1990	BSc ("Voriplom") in Mathematics.
	Oct 1988 until July 1990	Began to study Mathematics und Economics at the University of Göttingen, Germany.
Education:	May 1987	Finished school and passed the higher school certificate examination at my high school "Mariengymnasium Jever" in Jever, Lower Saxony, Germany.

Bremen, October 11th, 2004 Olaf Menkens